Tutorial 7: Graph Theory

Degrees and the Handshaking Theorem, complete graphs, walks and trails, Eulerian tours, bipartite and planar graphs

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. State and prove the **Handshaking Theorem** $\sum_v \deg(v) = 2|E|$ by counting the 1s of the **adjacency matrix** in two ways, and its directed version $\sum_v \deg^-(v) = \sum_v \deg^+(v) = |E|$.
2. Count the edges of the **complete graph** K_n, and prove by induction that K_n is $(n-1)$-regular.
3. Tell a **walk**, a **trail**, a **path**, a **circuit** and a **cycle** apart, and an **Eulerian** trail (every edge once) from a **Hamiltonian** path (every vertex once).
4. Prove **Euler's theorem** on Eulerian tours and trails, and use it to decide which graphs (for example which K_n) are Eulerian.
5. Prove that a connected graph on n vertices has at least $n-1$ edges.
6. Recognise **bipartite**, **complete bipartite** and **planar** graphs, and prove that K_5 and $K_{3,3}$ are not planar with **Euler's formula** $v - e + f = 2$.
7. Combine degree counting with a careful case analysis to force a **triangle** (a cycle of length 3).

Sections 1.1 to 1.7 review the tools with the intuition, a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 solves Problems 1 to 10 of the Graphs Tutorial 1 sheet in full. Section 3 collects the classic mistakes, "spot the error" puzzles, a checklist and exercises for self-study. The Graph Theory, Part 1 lecture (graphs, degree, K_n, connectedness) and the induction tutorials are assumed.

Conventions used throughout

Unless a problem says otherwise, a **graph** $G = (V, E)$ is finite, undirected, has **no loops** and **no multiple edges**. We write $u \sim v$ when $\{u, v\} \in E$. In a graph that does allow loops, a loop at v adds 2 to $\deg(v)$ (both of its ends are at v); in a directed graph it adds 1 to $\deg^-(v)$ and 1 to $\deg^+(v)$.

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1 The toolbox

This tutorial follows the first graph theory lecture. It starts with the one counting fact that is behind almost every argument about degrees (the Handshaking Theorem), uses it on complete graphs, and then moves to **travelling** in a graph: walks, trails and paths, and the question of Euler ("can I use every edge exactly once?"). It ends with two families of graphs, bipartite and planar, and the formula $v - e + f = 2$. The examples in this section are *not* the tutorial problems; those are solved in Section 2.

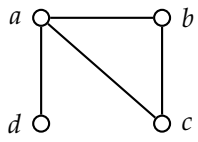
1.1 Degrees and the Handshaking Theorem

The intuition

At a party, every handshake involves **two** hands. If each guest reports how many hands they shook and we add the reports, every handshake is counted twice, once by each of the two people. So the total of the reports is **twice the number of handshakes**, and in particular it is **even**. In a graph: guests are vertices, handshakes are edges, and a report is a degree.
In a directed graph every arrow has **one tail and one head**. Counting tails gives the number of arrows, and so does counting heads.

Simple example: Figure 1 and a directed graph

In Figure 1, $3 + 2 + 2 + 1 = 8 = 2 \cdot 4$. ✓
For the directed graph with arcs (a, b), (b, c), (c, a), (a, c): out-degrees a:2, b:1, c:1 add to 4; in-degrees a:1, b:1, c:2 add to 4; and there are 4 arcs. ✓



deg: a:3, b:2, c:2, d:1

$$A = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ b & \mathbf{1} & 0 & \mathbf{1} & 0 \\ c & \mathbf{1} & \mathbf{1} & 0 & 0 \\ d & \mathbf{1} & 0 & 0 & 0 \end{array}$$

row sums: $3, 2, 2, 1$ (the degrees)
all entries: $8 = 2 \cdot 4$ edges
edge $\{a, b\}$ gives the two 1s at (a, b), (b, a)

Figure 1. A graph and its adjacency matrix. Adding the matrix by rows gives the sum of the degrees; adding it edge by edge gives two 1*s per edge. Same number, counted in two ways.*

Degree and the Handshaking Theorem

The **degree** $\deg(v)$ of a vertex v of an undirected graph is the number of edges incident to v. In a directed graph, the **in-degree** $\deg^-(v)$ is the number of arcs with v as their terminal vertex (head), and the **out-degree** $\deg^+(v)$ is the number with v as their initial vertex (tail).
Handshaking Theorem. For an undirected graph $G = (V, E)$, and for a directed graph $D = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2|E|, \qquad \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

Corollary. In every undirected graph the number of vertices of **odd** degree is **even**.

Comprehensive example: who shook how many hands?

(a) Can 5 people each shake hands with exactly 3 of the others? **(b)** A graph has 15 edges and every vertex has degree 3. How many vertices does it have?
Solution. (a) The handshake graph would have 5 vertices of degree 3, so $\sum \deg = 15$, which is odd. By the Handshaking Theorem the sum is $2|E|$, which is even. Contradiction, so **no**. (b) If there are n vertices, $3n = \sum \deg = 2 \cdot 15 = 30$, so $n = 10$. (Such a graph exists: the Petersen graph.)

Warning

Directed graphs are not "twice". Every arc is counted once among the out-degrees and once among the in-degrees, so $\sum \deg^+(v) = |E|$, not $2|E|$. Only the sum $\sum \left(\deg^-(v) + \deg^+(v)\right)$ equals $2|E|$.

Handshaking in one breath

- Undirected: $\sum \deg(v) = 2|E|$ (each edge has two ends). Directed: $\sum \deg^- = \sum \deg^+ = |E|$.
- So the degree sum is even, and the number of odd-degree vertices is even.

1.2 Complete graphs and regular graphs

The intuition

In the **complete graph** K_n, every pair of the n vertices is joined by an edge. So an edge is the same thing as **a choice of** 2 **vertices out of** n, and every vertex is joined to **all** $n - 1$ **others**. A graph in which every vertex has the same degree k is called k**-regular**; K_n is the most famous

example.

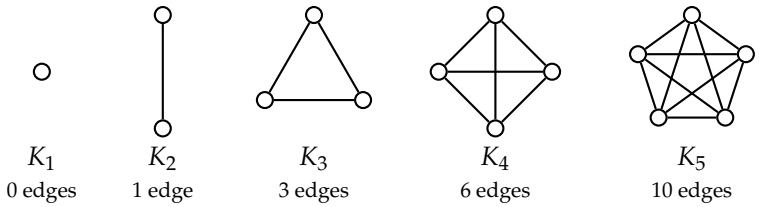


Figure 2. The complete graphs K_1 to K_5. Each vertex of K_n has degree $n-1$.

Simple example: K_4

K_4 has $\binom{4}{2} = 6$ edges, and each of its 4 vertices has degree 3. Handshaking check: $4 \cdot 3 = 12 = 2 \cdot 6$. ✓

Complete graphs and regular graphs

The **complete graph** K_n ($n \geq 1$) has n vertices and an edge between every two distinct vertices. A graph is **k-regular** if $\deg(v) = k$ for every vertex v. Then

$$|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}, \qquad K_n \text{ is } (n-1)\text{-regular},$$

and a k-regular graph on n vertices has $\frac{nk}{2}$ edges. The last formula is the Handshaking Theorem; it shows that nk must be even (no 3-regular graph on 7 vertices).

Comprehensive example: counting the edges of K_n by induction

Prove by induction that $|E(K_n)| = \frac{n(n-1)}{2}$ for all $n \geq 1$.
Solution. *Base case* ($n = 1$): K_1 has no edges, and $\frac{1 \cdot 0}{2} = 0$. ✓
Inductive hypothesis. Assume $|E(K_k)| = \frac{k(k-1)}{2}$ for some $k \geq 1$.
Inductive step. The vertices $v_1, \ldots, v_{k+1}$ of K_{k+1} split its edges into two disjoint groups: the edges among $v_1, \ldots, v_k$, which form a copy of K_k ($\frac{k(k-1)}{2}$ edges by the hypothesis), and the edges at v_{k+1}, one to each of $v_1, \ldots, v_k$ (k edges). By the rule of sum

$$|E(K_{k+1})| = \frac{k(k-1)}{2} + k = \frac{k(k-1)+2k}{2} = \frac{(k+1)k}{2}. \qquad \blacksquare$$

n vertices, degree $n-1$

A vertex is *not* adjacent to itself (no loops), so in K_n each vertex has $n-1$ neighbours, not n. Mixing up n and $n-1$ is the main source of wrong answers in the "for which n is K_n Eulerian" question.

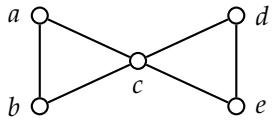
Complete graphs in one breath

- $|E(K_n)| = \binom{n}{2}$; every vertex has degree $n-1$.
- k-regular on n vertices: $\frac{nk}{2}$ edges, so nk is even.

1.3 Walks, trails, paths, circuits, cycles

The intuition

A **walk** is any trip along edges: you may reuse vertices and edges as often as you like. Every other word in this section is a walk with a **"no repeats"** rule: a **trail** never reuses an *edge*, and a **path** never reuses a *vertex* (so it cannot reuse an edge either). "Closed" means you end where you started: a closed trail is a **circuit** and a closed path is a **cycle**. Two special trips use *everything*: an **Eulerian** trail uses every *edge* exactly once, a **Hamiltonian** path visits every *vertex* exactly once.



the bowtie B

Sequence	edge repeated?	vertex repeated?	it is a ...
a, b, c, a, b	yes (ab)	yes	walk only
a, b, c, d, e, c	no	yes (c)	trail, not a path
a, b, c, d	no	no	path
a, b, c, d, e, c, a	no	yes (c)	circuit, not a cycle
c, d, e, c	no	only start = end	cycle (length 3)

Figure 3. Five trips in the bowtie B. The circuit a, b, c, d, e, c, a uses all six edges exactly once: it is an Eulerian tour (Section 1.4).

Simple example: lengths

The walk a, b, c, a, b has length 4 (four steps) and 5 entries. A walk of length k lists $k+1$ vertices $v_0, v_1, \ldots, v_k$. The cycle c, d, e, c has length 3: a **triangle**.

Walks and their relatives

Let $G = (V, E)$ be a graph. A **walk** of length k is a sequence $W = (v_0, v_1, \ldots, v_k)$ of vertices with $v_0 \sim v_1 \sim \cdots \sim v_k$. It is **closed** if $v_0 = v_k$.

Name	Rule	Closed version	Uses everything
walk	none	closed walk	
trail	no edge repeated	**circuit** (closed trail)	**Eulerian trail / tour**: every edge once
path	no vertex repeated	**cycle** (closed path)	**Hamiltonian path / cycle**: every vertex once

In a cycle the only repetition allowed is $v_0 = v_k$; in a graph without loops or multiple edges a cycle has length ≥ 3. A graph is **connected** if there is a walk (equivalently a path) between every two of its vertices.

Comprehensive example: Eulerian versus Hamiltonian

(a) The cube Q_3 (vertices: bit strings of length 3; edges: strings differing in one bit) has the Hamiltonian cycle

$$000,\ 001,\ 011,\ 010,\ 110,\ 111,\ 101,\ 100,\ 000,$$

since consecutive strings differ in exactly one bit and all 8 strings appear once. But Q_3 has no Eulerian trail: all 8 vertices have odd degree 3 (Section 1.4 explains why that rules it out).

(b) The bowtie B of Figure 3 has an Eulerian tour, but **no** Hamiltonian cycle: a cycle through a must use both edges ca and ab, and a cycle through d must use both cd and de, so it would contain 4 edges at c, while a cycle uses only 2 edges at each of its vertices.
So the two notions are independent: "every edge once" and "every vertex once" are different questions.

Path is the strictest word

Every path is a trail, and every trail is a walk, but not conversely. When a question asks for a *path*, check **vertices**; when it asks for a *trail*, check **edges**. (Some books use "path" for what we call a walk; in this course, a path never repeats a vertex.)

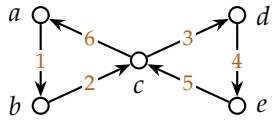
Walks in one breath

- Walk: anything. Trail: no repeated edge. Path: no repeated vertex. Closed trail = circuit, closed path = cycle.
- Eulerian: every *edge* once. Hamiltonian: every *vertex* once.

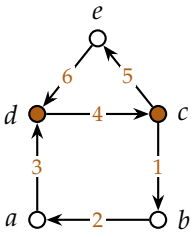
1.4 Eulerian trails and tours

The intuition

Imagine tracing a drawing without lifting the pen and without going over a line twice. Every time the pen passes *through* a vertex, it uses **two** edges there: one to come in, one to go out. So at every vertex other than the start and the end, the edges are used up **in pairs**: the degree must be even. If the drawing starts and ends at the same vertex, that vertex is also used in pairs (first exit with last entry). Euler's theorem says the converse also holds, for connected graphs: even degrees are *enough*.



bowtie: all degrees even
Eulerian **tour** from a



house: c, d odd (orange)
Eulerian **trail** from c to d

Figure 4. Left: the tour a, b, c, d, e, c, a. Right: the trail c, b, a, d, c, e, d. The numbers give the order in which the edges are used.

Simple example: the house

In the house graph of Figure 4 the degrees are a:2, b:2, c:3, d:3, e:2. Exactly two are odd, so an Eulerian trail must start at one of c, d and end at the other: c, b, a, d, c, e, d uses all 6 edges once. There is no Eulerian tour.

Eulerian trails and tours; Euler's theorem

An **Eulerian trail** in G is a walk that traverses every edge exactly once. If it also begins and ends at the same vertex it is an **Eulerian tour**, and a graph with an Eulerian tour is called **Eulerian**.
Euler's theorem. Let G be a connected graph.

(i) If every vertex of G has even degree, then for every vertex v there is an Eulerian tour that begins and ends at v.

(ii) If every vertex has even degree except two vertices a and b of odd degree, then G has an Eulerian trail that begins at a and ends at b.

Converse (the easy direction). If G has an Eulerian tour, every degree is even; if it has an Eulerian trail from a to $b \neq a$, then a and b are the only odd-degree vertices. So for a connected graph:

$$\text{Eulerian tour} \iff \text{0 odd vertices}, \qquad \text{Eulerian trail, not a tour} \iff \text{exactly 2 odd vertices}.$$

Comprehensive example: the bridges of Königsberg

The city of Königsberg had four land masses joined by seven bridges: land mass A had 5 bridges, and B, C, D had 3 each. Can one walk crossing every bridge exactly once?
Solution. Model land masses as vertices and bridges as edges (a multigraph: some pairs are joined by two bridges; the argument only uses degrees, so it still applies). The degrees $5, 3, 3, 3$ are all odd. A walk using every edge once is an Eulerian trail, and an Eulerian trail has at most 2 odd vertices (its two ends). Here there are 4, so **no such walk exists**. This was Euler's original argument (1736).

Warning

Do not forget "connected". Two separate triangles have all degrees equal to 2, but no Eulerian tour: a walk cannot jump from one triangle to the other. Even degrees are enough only for connected graphs (isolated vertices are harmless, since they have no edges to traverse).

Euler in one breath

- Connected, 0 odd vertices: Eulerian tour from any vertex. Connected, 2 odd vertices a, b: trail from a to b.
- 4 or more odd vertices: not even an Eulerian trail.

1.5 A small zoo of graphs

The intuition

A few families come back in every graph theory question. Knowing their vertex counts, edge counts and degrees by heart lets you test a conjecture in seconds: "is it true for C_5? for $K_{1,4}$? for Q_3?"

Simple example: checking a count with handshaking

Q_3 has 8 vertices of degree 3, so $|E| = \frac{8 \cdot 3}{2} = 12$. Count the segments in Figure 5: $4 + 4 + 4 = 12$. ✓

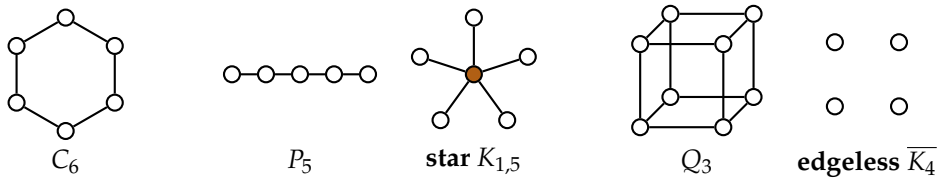


Figure 5. Some of the families seen so far (complete graphs are in Figure 2, complete bipartite graphs in Figure 6).

The families and their numbers

Graph	Description	vertices	edges	degrees
complete K_n	all pairs joined	n	$\binom{n}{2}$	all $n-1$
cycle (ring) C_n, $n \geq 3$	$v_1 \sim v_2 \sim \cdots \sim v_n \sim v_1$	n	n	all 2
path P_n	$v_1 \sim v_2 \sim \cdots \sim v_n$	n	$n-1$	two 1s, rest 2
star $K_{1,m}$	centre joined to m leaves	$m+1$	m	centre m, leaves 1
hypercube Q_n	bit strings, differ in one bit	2^n	$n2^{n-1}$	all n
edgeless $\overline{K_n}$	no edges at all	n	0	all 0
complete bipartite $K_{m,n}$	Section 1.6	$m+n$	mn	n and m (by side)

(Some books index paths by the number of edges instead; always say which convention you use.)

Comprehensive example: which of these are Eulerian?

Using the degree column and Euler's theorem (all are connected except $\overline{K_n}$ with $n \geq 2$): C_n is Eulerian (all degrees 2). P_n ($n \geq 2$) has exactly two odd vertices, its ends: Eulerian trail, no tour. $K_{1,m}$ with $m \geq 3$ has at least 3 odd leaves: neither. Q_n is Eulerian exactly when n is even.

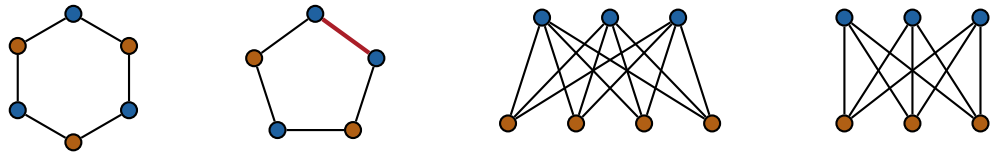
The zoo in one breath

- Remember vertices, edges and degrees of $K_n, C_n, P_n, K_{1,m}, Q_n, K_{m,n}$; check them with $\sum \deg = 2|E|$.
- Use the small members of each family to test conjectures before proving them.

1.6 Bipartite graphs

The intuition

A graph is **bipartite** if you can colour its vertices with two colours, say blue and orange, so that **every edge joins a blue vertex to an orange vertex**: no edge stays inside one colour. Think of students and courses, with an edge when a student takes a course. Walking along any edge *switches* colour, so a closed walk must switch an even number of times: **a bipartite graph has no cycle of odd length**, and in particular no triangle.



C_6: **bipartite** C_5: **not bipartite** $K_{3,4}$: 12 **edges** $K_{3,3}$: 9 **edges**

Figure 6. Blue and orange are the two parts. In C_5 the colours must alternate around the cycle, and after five steps the last edge (red) joins two blue vertices.

Simple example: even and odd cycles

C_6 is bipartite: colour the vertices alternately around the cycle. C_5 is not: going around, the colours must alternate, so v_1, v_3, v_5 get one colour, and then the edge v_5v_1 joins two vertices of the same colour.

Bipartite and complete bipartite graphs

A graph $G = (V, E)$ is **bipartite** if V can be partitioned into two sets V_1, V_2 ($V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$) so that every edge has the form $\{x, y\}$ with $x \in V_1$ and $y \in V_2$. The sets V_1, V_2 are the **parts**.
The **complete bipartite graph** $K_{m,n}$ has parts of sizes m and n, and an edge between two vertices **if and only if** they lie in different parts. It has $m + n$ vertices and mn edges (rule of product: one end in each part).
Fact. Every cycle in a bipartite graph has even length. (The converse is also true: a graph with no odd cycle is bipartite.)

Comprehensive example: the hypercube is bipartite

Show that Q_n is bipartite for every $n \geq 1$.
Solution. Let V_1 be the bit strings with an **even** number of 1s and V_2 those with an **odd** number. These sets partition the vertex set. Two adjacent strings differ in exactly one bit, so their numbers of 1s differ by exactly 1: one is even and one is odd. Hence every edge joins V_1 to V_2, and Q_n is bipartite. ■
Consequence. Q_n has no triangle (no odd cycle at all). Check in Q_3: $V_1 = \{000, 011, 101, 110\}$. ✓

"Bipartite" is not "complete bipartite"

A bipartite graph only forbids edges *inside* a part; it does not require all edges *between* the parts. C_6 is bipartite but not complete bipartite. Also, $K_{m,n}$ has mn edges, not $\binom{m+n}{2}$.

Bipartite graphs in one breath

- Two parts, every edge goes across. Equivalently: a proper 2-colouring. No odd cycles, so no triangles.
- $K_{m,n}$: all mn cross edges; degrees n on the m-side and m on the n-side.

1.7 Planar graphs and Euler's formula

The intuition

Some graphs can be drawn on paper with no two edges crossing. Such a drawing cuts the plane into regions called **faces** (count the outside region too). Euler noticed that the three numbers v (vertices), e (edges) and f (faces) always satisfy $v - e + f = 2$. Since each face needs at least 3 edges around it, a planar graph cannot have too many edges, and that is how we prove that a graph is *not* planar: show it has too many edges.

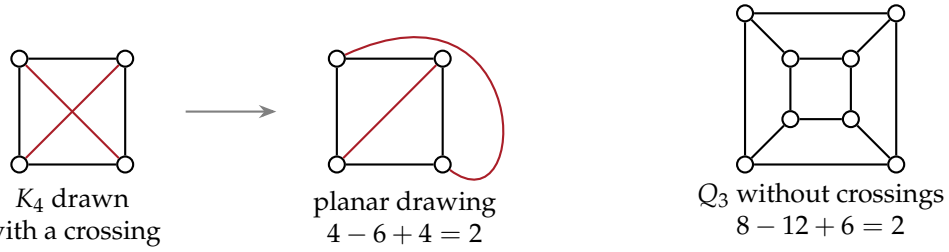


Figure 7. Left: K_4 is planar, because it can *be redrawn without crossings (move one red edge outside). Right: the cube Q_3 as a square inside a square, with* 6 *faces (the inner square, four trapezoids, the outside).*

Simple example: counting faces of K_4

In the planar drawing of K_4 in Figure 7 there are 3 bounded triangular regions and the outside region: $f = 4$. With $v = 4$ and $e = 6$: $v - e + f = 4 - 6 + 4 = 2$. ✓

Planar graphs, Euler's formula and its consequences

A graph is **planar** if it can be drawn in the plane so that edges meet only at their common endpoints (such a drawing is a **planar embedding**). A planar embedding divides the plane into **faces** (regions), including the unbounded one.

Euler's formula. For a connected planar graph with v vertices and e edges, every planar embedding has

$$v - e + f = 2 \qquad \text{faces } f.$$

Edge bounds. Let G be connected, planar, without loops or multiple edges, with $v \geq 3$. Then

$$e \leq 3v - 6, \qquad \text{and if } G \text{ has no triangle (for example } G \text{ bipartite)}: \quad e \leq 2v - 4.$$

Why. Walk around the boundary of every face and add the lengths: each edge is counted exactly twice (once from each side), so the total is $2e$. Each face has boundary length at least 3 (at least 4 without triangles), so $3f \leq 2e$ (resp. $4f \leq 2e$). Put $f \leq \frac{2e}{3}$ into $2 = v - e + f$: $2 \leq v - \frac{e}{3}$, that is $e \leq 3v - 6$. With $f \leq \frac{e}{2}$ the same steps give $e \leq 2v - 4$.

Kuratowski's theorem (statement only). A graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or of $K_{3,3}$.

Comprehensive example: a planar graph has a vertex of degree at most 5

Let G be connected, planar, without loops or multiple edges, with $v \geq 3$. Then $\sum \deg(x) = 2e \leq 2(3v-6) = 6v-12 < 6v$. If every vertex had degree at least 6, the sum would be at least $6v$. So some vertex has degree ≤ 5. (This fact is the starting point of the proof that planar graphs can be coloured with 6 colours.)

Planar means *can* be drawn

A drawing with crossings proves nothing: K_4 in Figure 7 is planar although the first drawing has a crossing. To show a graph is planar, exhibit one drawing without crossings; to show it is *not* planar, you need an argument that rules out *every* drawing, such as an edge bound.

Planarity in one breath

- Connected planar: $v - e + f = 2$. Simple, $v \geq 3$: $e \leq 3v - 6$; no triangles: $e \leq 2v - 4$.
- Too many edges $\Rightarrow$ not planar. K_5 and $K_{3,3}$ are the two basic non-planar graphs.

The toolbox at a glance

Situation	Tool	Signal words
Sum of degrees, parity	$\sum \deg = 2\|E\|$; odd vertices come in pairs	"degree", "number of edges", "odd"
Complete graph	$\binom{n}{2}$ edges, $(n-1)$-regular	"K_n", "every pair"
Trip through the graph	check edges (trail) or vertices (path)	"walk", "trail", "path", "cycle"
Every edge once	Euler: count odd vertices, check connected	"Eulerian", "without lifting the pen"
Every vertex once	construct it, or find an obstruction	"Hamiltonian"
Two classes, edges across	2-colouring; no odd cycles	"bipartite", "$K_{m,n}$"
Drawing without crossings	$v - e + f = 2$; $e \leq 3v-6$, $e \leq 2v-4$	"planar", "embedding"

2 The tutorial problems, worked in full

2.1 How to write the solutions

Most problems this week are **proofs**. A proof about graphs should name the graph and its vertex and edge sets, say which theorem or definition each step uses, and check the small cases (the base case, $n = 1$ or $n = 2$, a vertex of degree 0 or 1). As before, write the **Statement** first and then the **Solution**.

Writing template for a counting proof about a graph ("count one thing two ways")

Statement. *[The equality to prove, for which graphs.]*
The quantity. *["We count the number of ... in two ways." For example: the* 1*s of the adjacency matrix, the pairs (vertex, incident edge).]*
First count. *[Grouped by vertices.]* **Second count.** *[Grouped by edges.]*
Conclusion. *["Both count the same quantity, so ..." ■]*

Writing template for an induction on the number of vertices or edges

Statement. *["For every $n \geq n_0$, every graph with n vertices and property P has ..."]*
Base case. *[The smallest graph; check the claim directly.]*
Inductive hypothesis. *["Assume the statement holds for all graphs with k vertices" (or "with fewer than m edges" for strong induction).]*
Inductive step. *[Start from an **arbitrary** graph G with $k+1$ vertices. Remove something to get a smaller graph G', **check that** G' **still has property** P, apply the hypothesis to G', then put the removed part back.]*
Conclusion. *["By induction, the statement holds for all $n \geq n_0$." ■]*

Phrasebook: new sentences for this tutorial

Where	What to write
Handshaking	"By the Handshaking Theorem, $\sum_{v \in V} \deg(v) = 2\lvert E\rvert$, which is even."
Odd vertices	"The number of vertices of odd degree is even, so it cannot be 1."
Removing a vertex	"Let $G - v$ be the graph obtained by deleting v and the edges incident to it."
Keeping the hypothesis	"$G - v$ is still connected, because no path between two vertices other than v uses v."
Euler	"G is connected and every vertex has even degree, so by Euler's theorem G has an Eulerian tour."
Planarity	"Suppose, for a contradiction, that K_5 is planar. Then $e \leq 3v - 6$, so $10 \leq 9$, which is false."
Bipartite	"Every edge joins a vertex with an even number of 1s to one with an odd number."

2.2 Problem 1: the Handshaking Theorem (adjacency matrix proof)

Problem 1

Let $G = (V, E)$ be an undirected graph. Prove that $\sum_{v \in V} \deg(v) = 2|E|$ by a combinatorial argument using the adjacency matrix. Then state and prove the version for directed graphs: $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$.

How to think about it. Section 1.1 and Figure 1: add up all entries of the adjacency matrix in two ways.

Model solution to Problem 1

Solution (what you write)	Commentary (what it does)
Set-up. Let $V = \{v_1, \ldots, v_n\}$ and let $A = (a_{ij})$ be the $n \times n$ **adjacency matrix**: $a_{ij} = 1$ if $\{v_i, v_j\} \in E$ and $a_{ij} = 0$ otherwise. We count $S = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$, the number of 1s in A, in two ways.	*Name the quantity once; both counts refer to it.*
First count (by rows). Row i has a 1 in column j exactly when v_j is a neighbour of v_i, so $\sum_j a_{ij} = \deg(v_i)$. Adding over the rows: $S = \sum_{i=1}^{n} \deg(v_i)$.	*Row sums are degrees.*

Second count (by edges). An edge $\{v_i, v_j\}$ (with $i \neq j$, since there are no loops) produces exactly two 1s, at positions (i, j) and (j, i), and every 1 in A is produced by exactly one edge. So $S = 2|E|$.

A is symmetric with zeros on the diagonal.

Conclusion. Both expressions count the 1s of A, so $\sum_{v \in V} \deg(v) = 2|E|$. ■

Directed version. For a directed graph let $a_{ij} = 1$ if (v_i, v_j) is an arc. Row i has a 1 for each arc *leaving* v_i, so its sum is $\deg^+(v_i)$; column j has a 1 for each arc *entering* v_j, so its sum is $\deg^-(v_j)$. Each arc gives exactly **one** 1. Adding all entries by rows, by columns and by arcs:

Not symmetric any more: one entry per arc. A loop (v_i, v_i) is the diagonal entry a_{ii}; it counts once in row i and once in column i, which matches "a loop adds 1 to each of $\deg^-$ and $\deg^+$".

$$\sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v) = |E|. \qquad \blacksquare$$

Check (Figure 1): rows $3 + 2 + 2 + 1 = 8$; edges $2 \cdot 4 = 8$. ✓

Another way to see it

The same proof without a matrix. Count the pairs (v, e) with v a vertex, e an edge, and v an end of e. Grouped by v: $\sum \deg(v)$. Grouped by e: each edge has 2 ends, $2|E|$. The adjacency matrix is just a neat table of these pairs.

2.3 Problem 2: the number of edges of K_n

Problem 2

How many edges are in K_n, the complete graph on n vertices?

Model solution to Problem 2

Solution (what you write) | **Commentary (what it does)**

Solution. In K_n every two distinct vertices are joined by exactly one edge, and every edge joins two distinct vertices. So the edges of K_n correspond one-to-one to the 2-element subsets of the vertex set:

An edge is an unordered *pair, so $\binom{n}{2}$, not $n(n-1)$.*

$$|E(K_n)| = \binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \boxed{\frac{n(n-1)}{2}}.$$

Second way. Every vertex of K_n has degree $n - 1$, so by the Handshaking Theorem $2|E| = n(n-1)$ and $|E| = \frac{n(n-1)}{2}$.

Both counts agree, as they must.

Check. K_4: 6 edges; K_5: 10 edges (Figure 2). ✓

2.4 Problem 3: K_n is $(n-1)$-regular, by induction

Problem 3

Give an inductive proof that K_n is $(n-1)$-regular for every $n \geq 1$.

How to think about it. Same splitting as the comprehensive example of Section 1.2: K_{k+1} is a copy of K_k plus one new vertex joined to everything. There are two kinds of vertices to check: the old ones and the new one.

Model solution to Problem 3

Solution (what you write)	Commentary (what it does)
Statement. For every $n \geq 1$, every vertex of K_n has degree $n-1$.	*k-regular means every degree equals k.*
Base case ($n = 1$). K_1 has one vertex and no edges, so its vertex has degree $0 = 1 - 1$. ✓	*Starting at $n = 2$ is also fine: K_2 is one edge, both degrees $1 = 2 - 1$.*
Inductive hypothesis. Assume that for some $k \geq 1$, every vertex of K_k has degree $k-1$.	
Inductive step. Let the vertices of K_{k+1} be $v_1, \ldots, v_{k+1}$. Deleting v_{k+1} leaves the graph on $v_1, \ldots, v_k$ in which every two vertices are still adjacent: a copy of K_k. *Old vertices.* For $i \leq k$, by the hypothesis v_i has $k-1$ neighbours among $v_1, \ldots, v_k$; in K_{k+1} it has one more neighbour, v_{k+1}. So $\deg(v_i) = (k-1) + 1 = k$. *New vertex.* v_{k+1} is adjacent to each of $v_1, \ldots, v_k$ and to nothing else, so $\deg(v_{k+1}) = k$. Hence every vertex of K_{k+1} has degree $k = (k+1) - 1$.	*Both kinds of vertices must be checked. Forgetting the new vertex is the classic gap.*
Conclusion. By induction, K_n is $(n-1)$-regular for all $n \geq 1$. ■	

2.5 Problem 4: examples of Eulerian and Hamiltonian trips

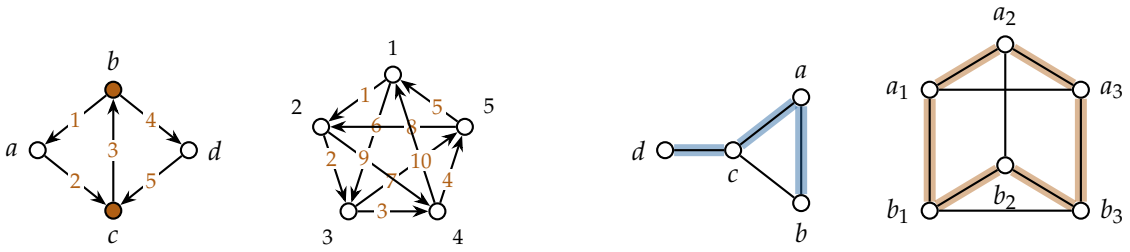
Problem 4

(a) Draw an example of an Eulerian trail (that is not a tour) and of an Eulerian tour.
(b) Draw an example of a Hamiltonian path (in a graph with no Hamiltonian cycle) and of a Hamiltonian cycle. Also give a graph with no Hamiltonian path at all.

How to think about it. Eulerian: *don't repeat edges*, and use all of them. Hamiltonian: *don't repeat vertices*, and visit all of them. For the Eulerian examples, choose graphs with 2 (trail) or 0 (tour) odd vertices.

Model solution to Problem 4

Solution (what you write)	Commentary (what it does)



(a) Eulerian trail b, a, c, b, d, c

(a) Eulerian tour of K_5 $1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 1$

(b) Ham. path d, c, a, b; no Ham. cycle

(b) Ham. cycle $a_1, a_2, a_3, b_3, b_2, b_1, a_1$

Figure 8. Model answers for Problem 4. Orange vertices have odd degree. Highlighted edges form the Hamiltonian path (blue) and the Hamiltonian cycle (orange) on the triangular prism.

(a) Eulerian trail. The *diamond* (K_4 minus one edge) has vertices a, b, c, d and edges ab, ac, bc, bd, cd. Degrees: a:2, b:3, c:3, d:2. The walk b, a, c, b, d, c uses the 5 edges ba, ac, cb, bd, dc, each exactly once, and starts and ends at different vertices.

Two odd vertices (b, c): the trail must go from one to the other. There is no tour.

(a) Eulerian tour. K_5 (all degrees 4). The closed walk $1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 1$ uses the edges $12, 23, 34, 45, 51$ (the outer pentagon) and $13, 35, 52, 24, 41$ (the inner star): all 10 edges, each once.

Outer cycle, then the "star" cycle: two edge-disjoint cycles through vertex 1 glued together.

(b) Hamiltonian path, no cycle. A triangle a, b, c with an extra vertex d joined only to c. The path d, c, a, b visits all four vertices once. There is no Hamiltonian cycle: $\deg(d) = 1$, but a cycle uses 2 edges at each of its vertices.

A degree-1 vertex kills every Hamiltonian cycle.

(b) Hamiltonian cycle. The triangular prism: triangles $a_1a_2a_3$, $b_1b_2b_3$ and the edges a_ib_i. The cycle $a_1, a_2, a_3, b_3, b_2, b_1, a_1$ visits all six vertices once (Figure 8).

The cube Q_3 of Section 1.3 is another good answer.

No Hamiltonian path. The star $K_{1,3}$: a path visits the centre at most once, so it contains at most 2 of the 3 leaves (one on each side of the centre).

A path through the centre can go out to only two leaves.

2.6 Problem 5: proof of Euler's theorem

Problem 5

Prove Euler's theorem (Section 1.4): (a) a connected graph with all degrees even has an Eulerian tour starting and ending at any given vertex v; (b) a connected graph whose only odd vertices are $a \neq b$ has an Eulerian trail from a to b; (c) conversely, the degree conditions are necessary.

How to think about it. Key idea for (a): if all degrees are even, then a trail that starts at v **cannot get stuck anywhere except at** v: whenever you enter another vertex, you have used an odd number of its edges, so an unused one is left to leave by. So you can always walk a closed trail C. If C misses some edges, the leftover pieces still have even degrees, so (by induction) they have their own tours, which can be **spliced** into C.

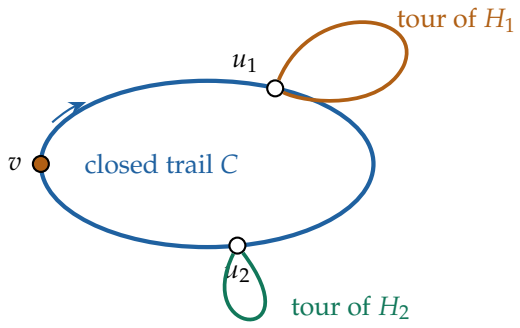


Figure 9. Splicing. Follow C from v; at u_1 take the detour around H_1 and come back to u_1; continue along C; at u_2 take the detour around H_2; finish at v.

Model solution to Problem 5(a)

Solution (what you write)	**Commentary (what it does)**
Statement. If G is connected and every vertex has even degree, then for every vertex v there is an Eulerian tour of G beginning and ending at v.	*Graphs may have multiple edges here; the proof works without change.*
Proof by strong induction on $m = \|E\|$. *Base case* ($m = 0$). A connected graph with no edges has a single vertex v, and the walk (v) of length 0 is an Eulerian tour.	*There are no edges to use, so the empty tour uses all of them.*
Inductive hypothesis. Let $m \geq 1$ and assume the statement for every connected graph with all degrees even and fewer than m edges.	*Strong induction: the leftover pieces can have any smaller number of edges.*
Step 1: a closed trail through v. Let G have m edges. Since G is connected and has an edge, it has at least 2 vertices, so no vertex has degree 0; being even, every degree is at least 2. Start at v and walk along unused edges for as long as possible. This is a trail, and it stops because G is finite. It cannot stop at a vertex $w \neq v$: each earlier pass through w used 2 of its edges, and the final arrival used 1, so an odd number of w's edges is used; as $\deg(w)$ is even, an unused edge at w remains. Hence the trail stops at v: it is a circuit C through v of positive length.	*This is the "key idea": with even degrees you never get stuck, except back home.*
Step 2: the leftover graph. Let G' be G with the edges of C removed. At each vertex, C uses an even number of edges (in and out in pairs), so every degree in G' is still even. Let $H_1, \ldots, H_r$ be the connected components of G' that have at least one edge. Each H_i is connected, has all degrees even, and has fewer than m edges. By the hypothesis, H_i has an Eulerian tour beginning and ending at any of its vertices.	*Isolated vertices of G' have nothing to traverse; ignore them.*

Step 3: each H_i touches C. Pick a vertex x of H_i and a path P in G from x to v (G is connected). Let u_i be the first vertex of P that lies on C. The edges of P before u_i have an end off C, so they are not edges of C: they lie in G'. Hence u_i is in the same component as x, that is $u_i \in H_i$.

This is where connectedness of G is used.

Step 4: splice. Follow C from v. The first time we reach a vertex u_i of a component H_i not yet used, insert the Eulerian tour of H_i that begins and ends at u_i, then continue along C. The result is a closed walk from v to v. It uses every edge of C once and every edge of each H_i once, and these edge sets partition E. So it is an Eulerian tour of G from v. ■

Figure 9. The components are edge-disjoint from each other and from C.

Model solution to Problem 5(b) and (c)

Solution (what you write)	Commentary (what it does)
(b) Let G be connected with exactly two odd vertices $a \neq b$. Build G^+ by adding a **new** vertex x and the two edges xa, xb. In G^+: a and b gain 1 and become even, x has degree 2, all other degrees are unchanged (even), and G^+ is connected. By (a), G^+ has an Eulerian tour beginning and ending at x. It leaves x along one of xa, xb and returns along the other; reversing the tour if needed, it is $x, a, \ldots, b, x$. Deleting the first and last step leaves a walk from a to b that uses every edge of G exactly once: an Eulerian trail from a to b. ■	*Reduce (b) to (a). (Adding the edge ab directly also works if multiple edges are allowed.)*
(c) Necessity. Suppose W is an Eulerian trail. Each time W passes *through* a vertex it uses two of its edges; the first vertex of W also uses one edge to leave, and the last one edge to arrive. Since every edge is used exactly once, each vertex's degree is the number of edges W uses there. If W is a *tour* (start = end), the first exit and the last arrival pair up, so every degree is even. If W goes from a to $b \neq a$, then a and b have odd degree (one unpaired edge each) and every other vertex has even degree. ■	*This is the "check the degree conditions" part: in, out, in, out.*

2.7 Problem 6: why exactly two odd vertices?

Problem 6

Why does Euler's theorem talk about *zero* or *two* odd vertices only? Can a graph have exactly one vertex of odd degree?

Model solution to Problem 6

Solution (what you write)	Commentary (what it does)
Claim. In every graph the number of odd-degree vertices is even. In particular no graph has exactly one (or three, or five) odd vertices.	*The corollary of Section 1.1.*

Proof. Split V into V_{even} and V_{odd} by the parity of the degree. By the Handshaking Theorem

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = 2|E|.$$

Even − even = even.

The right side and the first sum are even, so $\sum_{v \in V_{\text{odd}}} \deg(v)$ is even. It is a sum of $|V_{\text{odd}}|$ odd numbers, and a sum of odd numbers is even only when there is an even number of terms. So $|V_{\text{odd}}|$ is even. ■

Why only 0 or 2. So the possible numbers of odd vertices are $0, 2, 4, 6, \ldots$ If there are 4 or more, there is no Eulerian trail at all: by Problem 5(c), only the two ends of a trail can have odd degree. So the cases 0 (tour) and 2 (trail) are the only ones in which an Eulerian trail can exist.

One odd vertex is impossible, and ≥ 4 is hopeless.

2.8 Problem 7: for which n is K_n Eulerian?

Problem 7

For which values of n is the complete graph K_n Eulerian?

Model solution to Problem 7

Solution (what you write)	Commentary (what it does)
Degrees. K_n is connected and $(n-1)$-regular (Problem 3).	*Degree $n-1$, not n.*
n **odd.** Then $n-1$ is even, so every vertex has even degree, and by Euler's theorem K_n has an Eulerian tour. (For $n = 1$ the tour is the trivial one of length 0.)	*K_3: the triangle. K_5: Problem 4.*
n **even.** Then every one of the n vertices has odd degree $n-1$. By Problem 5(c) a tour needs 0 odd vertices, so K_n is not Eulerian. (For $n = 2$ there is an Eulerian *trail*, the single edge; for $n \geq 4$ there are at least 4 odd vertices and not even a trail.)	
Answer. K_n is Eulerian $\iff$ $\boxed{n \text{ is odd}}$.	*Checked by computer for $n \leq 8$.*

2.9 Problem 8: a connected graph on n vertices has at least $n-1$ edges

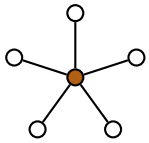
Problem 8

Show that if G is a connected graph on n vertices, then it has at least $n-1$ edges.

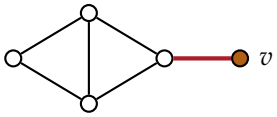
(The tutorial sheet says "on $n-1$ vertices"; that is a typo: the statement is about n vertices.)

How to think about it. Induction on n, removing one vertex. The danger: deleting a vertex can **disconnect** the graph (delete the centre of a star!), and then the hypothesis does not apply. So choose the vertex carefully: a vertex of degree 1 is always safe to delete. And if there is no vertex of degree 1,

handshaking alone finishes the job.



deleting the centre
leaves 5 isolated vertices



deleting a degree-1 vertex v
keeps the rest connected

Figure 10. Which vertex to remove in the inductive step.

Model solution to Problem 8

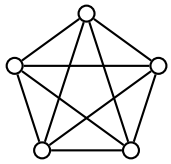
Solution (what you write)	**Commentary (what it does)**
Statement. For every $n \geq 1$: if G is a connected graph on n vertices, then $\lvert E(G)\rvert \geq n-1$.	
Base case ($n = 1$). Any graph has $\lvert E\rvert \geq 0 = 1-1$. ✓	
Inductive hypothesis. Assume that for some $n \geq 1$, every connected graph on n vertices has at least $n-1$ edges.	
Inductive step. Let $G = (V, E)$ be an **arbitrary** connected graph on $n+1 \geq 2$ vertices. We show $\lvert E\rvert \geq n$.	*Start from the big graph, not by building it from a small one.*
Case 1: every vertex has degree ≥ 2. By the Handshaking Theorem $2\lvert E\rvert = \sum_{v\in V} \deg(v) \geq 2(n+1)$, so $\lvert E\rvert \geq n+1 \geq n$.	*No induction needed in this case.*
Case 2: some vertex v has degree < 2. It cannot have degree 0: it would be isolated in a graph with $n+1 \geq 2$ vertices, contradicting connectedness. So $\deg(v) = 1$. Let $G - v$ be G with v and its single edge deleted. $G - v$ is connected: a path between two vertices other than v never passes through v (it would have to enter and leave v along two different edges, but v has only one), so it survives in $G - v$. Thus $G - v$ is a connected graph on n vertices, and by the hypothesis it has at least $n-1$ edges. Since G has exactly one more edge, $\lvert E\rvert \geq (n-1)+1 = n$.	*The point of choosing a degree-1 vertex: the hypothesis "connected" still holds for $G - v$.*
Conclusion. By induction, every connected graph on n vertices has at least $n-1$ edges. ■	*Equality holds exactly for trees, e.g. P_n and $K_{1,n-1}$.*

2.10 Problem 9: K_5 and $K_{3,3}$ are not planar

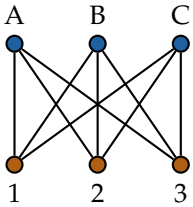
Problem 9

Discuss the planarity of K_5 and $K_{3,3}$: prove that neither is planar.

How to think about it. Section 1.7: count v and e, compare with the edge bound. For $K_{3,3}$ the general bound $3v - 6$ is **not** strong enough; use that it has no triangles.



K_5: $v = 5$, $e = 10$
bound $3v - 6 = 9$



$K_{3,3}$: $v = 6$, $e = 9$, no triangles
bound $2v - 4 = 8$

Figure 11. The two basic non-planar graphs. $K_{3,3}$ is the "three houses, three utilities" puzzle: houses A, B, C, utilities $1, 2, 3$.

Model solution to Problem 9

Solution (what you write)	**Commentary (what it does)**
(a) K_5**.** K_5 has $v = 5$ vertices and $e = \binom{5}{2} = 10$ edges, and it is connected, without loops or multiple edges. Suppose, for a contradiction, that K_5 is planar. Since $v \geq 3$, the edge bound gives $e \leq 3v - 6 = 9$. But $e = 10 > 9$. Contradiction, so K_5 is **not planar**. ■	*Equivalently: Euler's formula would force $f = 2 - 5 + 10 = 7$ faces, needing $3f = 21 > 2e = 20$ edge-sides.*
(b) $K_{3,3}$**.** $K_{3,3}$ has $v = 6$, $e = 3 \cdot 3 = 9$, and is connected. It is bipartite, so every cycle has even length; in particular it has no triangle. Suppose, for a contradiction, that $K_{3,3}$ is planar. Then the triangle-free bound gives $e \leq 2v - 4 = 8$. But $e = 9 > 8$. Contradiction, so $K_{3,3}$ is **not planar**. ■	*Euler's formula would force $f = 2 - 6 + 9 = 5$; each face needs ≥ 4 edge-sides, so $20 \leq 2e = 18$: impossible.*
Remark. The bound $e \leq 3v - 6 = 12$ is *satisfied* by $K_{3,3}$, so it proves nothing there; the stronger bound uses the absence of triangles. By Kuratowski's theorem, K_5 and $K_{3,3}$ are, in a precise sense, the *only* obstructions to planarity.	*A bound that holds proves nothing: you need a bound that* fails.

2.11 Problem 10: a 4-regular graph on 9 vertices contains a triangle

Problem 10

Prove that if a graph has 9 vertices and the degree of each vertex is 4, then it contains a cycle of length 3.

How to think about it. Assume there is no triangle and look at one edge uv. A triangle-free graph has two simple rules: **(1) the neighbours of a vertex are pairwise non-adjacent**, and **(2) two adjacent vertices have no common neighbour**. With degree 4, the neighbourhoods of u and v fill 8 of the 9 vertices. The ninth vertex w is the key: counting edges shows it must have neighbours on both sides, and that creates a triangle.

Why 9 matters. On 8 vertices the statement is false: $K_{4,4}$ is 4-regular, and bipartite, so it has no triangle. And counting edges alone cannot work: the graph has $\frac{9 \cdot 4}{2} = 18$ edges, while a triangle-free graph on 9 vertices may have up to $\lfloor 81/4 \rfloor = 20$ (Mantel's theorem). The structure, not just the number of edges, gives the triangle.

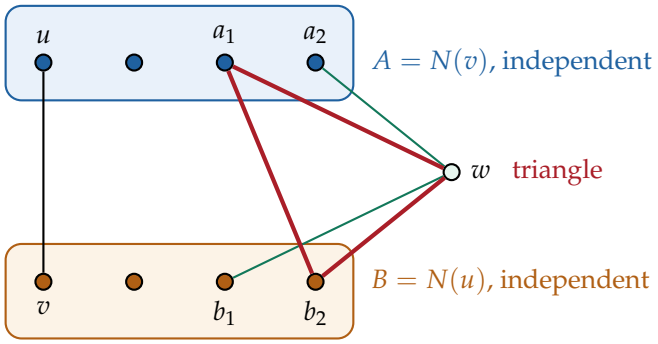


Figure 12. The structure forced by an edge uv in a triangle-free 4-regular graph on 9 vertices. Since a_1 misses only one vertex of B, it is adjacent to b_1 or b_2, and that closes a triangle with w.

Model solution to Problem 10

Solution (what you write)	**Commentary (what it does)**
Statement. Let G be a graph (no loops, no multiple edges) with 9 vertices, all of degree 4. Then G contains a cycle of length 3.	
Proof by contradiction. Suppose G has no triangle. Pick an edge uv (one exists, since degrees are 4). Let $A = N(v)$ and $B = N(u)$ be the neighbourhoods, so $u \in A$, $v \in B$, $\lvert A\rvert = \lvert B\rvert = 4$.	*$N(x)$ is the set of neighbours of x.*
Three facts. (i) A is independent (no edges inside): an edge xy with $x, y \in A$ would give the triangle v, x, y. Likewise B is independent. (ii) $A \cap B = \emptyset$: a common neighbour x of u and v would give the triangle u, v, x. (iii) So $\lvert A \cup B\rvert = 8$, and there is exactly one more vertex $w \notin A \cup B$.	*The two rules of triangle-free graphs.*
Where do the edges at A go? A vertex of A has no neighbour in A (fact (i)), so all 4 of its neighbours lie in $B \cup \{w\}$. Adding over A: $16 = e(A, B) + e(A, w)$, where $e(A, B)$ is the number of edges between A and B and $e(A, w)$ the number of neighbours of w in A. In the same way $16 = e(A, B) + e(B, w)$.	*Degree counting on one side, like handshaking.*
The vertex w. Subtracting, $e(A, w) = e(B, w)$. All 4 neighbours of w lie in $A \cup B$, so $e(A, w) + e(B, w) = 4$, hence $e(A, w) = e(B, w) = 2$. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ be the neighbours of w.	*w has two neighbours on each side.*
The triangle. The vertex a_1 has 4 neighbours in $B \cup \{w\}$, one of which is w; so it has exactly 3 neighbours in B, and misses exactly one of the 4 vertices of B. Therefore a_1 is adjacent to at least one of b_1, b_2, say b_j. Then $w \sim a_1$, $a_1 \sim b_j$, $b_j \sim w$: the triangle w, a_1, b_j, w. This contradicts the assumption, so G contains a cycle of length 3. ■	*Figure 12. The count "3 out of 4" is where $9 = 4 + 4 + 1$ is used.*
Check. A computer search over all graphs on 9 vertices finds no triangle-free 4-regular one, while on 8 vertices $K_{4,4}$ is one. ✓	

3 Mistakes, checklist and exercises

3.1 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **"$2|E|$" in a directed graph.** $\sum \deg^+(v) = |E|$; each arc has one tail.
2. **Forgetting the new vertex** in an induction that builds K_{k+1} from K_k (Problem 3).
3. **Trail versus path.** A trail may repeat vertices; a path may not.
4. **Euler without connectedness.** Two disjoint triangles have even degrees but no Eulerian tour.
5. **n versus $n-1$.** In K_n the degree is $n-1$: K_n is Eulerian for *odd* n.
6. **Deleting a vertex that disconnects the graph** in an induction (Problem 8). Check the hypothesis still applies.
7. **Using a bound that holds.** $K_{3,3}$ satisfies $e \leq 3v-6$; only the triangle-free bound $e \leq 2v-4$ fails.
8. **Circular planarity arguments.** "K_5 is not planar by Kuratowski's theorem" uses a theorem whose statement is built on K_5 being non-planar; give the edge count argument.
9. **Counting edges and stopping** (Problem 10). $18 \leq 20$ is no contradiction; the structure gives the triangle.

Spot the error 1: Problem 1 for directed graphs

"Solution." Every arc has two ends, so as in the undirected case $\sum_v \deg^+(v) = 2|E|$.
Where is the mistake? (Answer in Appendix A.)

Spot the error 2: Problem 3

"Inductive step." Assume K_k is $(k-1)$-regular. K_{k+1} is K_k with one extra vertex joined to all others, so each degree goes up by 1, and K_{k+1} is k-regular.
Where is the gap? (Answer in Appendix A.)

Spot the error 3: Euler's theorem

"Claim." The graph consisting of two disjoint triangles has all degrees equal to 2, so by Euler's theorem it has an Eulerian tour.
Where is the mistake? (Answer in Appendix A.)

Spot the error 4: Problem 8

"Inductive step." Take a connected graph with $n+1$ vertices. Remove **any** vertex together with its edges. The remaining graph has n vertices, so by the hypothesis it has at least $n-1$ edges. Putting the vertex back needs at least one edge to reconnect it, so the graph has at least $(n-1)+1 = n$ edges.
Where is the mistake? (Answer in Appendix A.) Hint: try it on the star $K_{1,5}$.

Spot the error 5: Problem 9(b)

"Solution." $K_{3,3}$ has $v = 6$ and $e = 9$. For a planar graph $e \leq 3v-6 = 12$, and $9 \leq 12$, so $K_{3,3}$ is planar.

Where is the mistake? (Answer in Appendix A.)

Spot the error 6: Problem 10

"Proof." Suppose G is triangle-free. The degree sum is 36, so $e = 18$. By Mantel's theorem a triangle-free graph on 9 vertices has at most $\lfloor 81/4 \rfloor = 20$ edges. With every vertex of degree 4 the graph is very dense, so it is impossible to avoid a triangle.
Where is the mistake? (Answer in Appendix A.)

3.2 Checklist before you hand in

Checklist

- ☐ I said whether the graph is directed or undirected, and whether loops or multiple edges are allowed.
- ☐ Degree sums: $2|E|$ for undirected graphs, $|E|$ for each of $\sum \deg^-$, $\sum \deg^+$.
- ☐ For walks, I checked **edges** (trail) or **vertices** (path) as the definition requires.
- ☐ Before using Euler's theorem I checked **connected** and counted the odd vertices.
- ☐ In every induction, the smaller graph **still satisfies the hypothesis** (connected, even degrees, ...).
- ☐ In the inductive step I started from an **arbitrary** graph of the larger size.
- ☐ For non-planarity I used a bound that **fails**, and I stated why the bound applies (simple, connected, $v \geq 3$, triangle-free if needed).
- ☐ My small cases (K_1, K_2, a vertex of degree 0 or 1) were checked.
- ☐ Every proof ends with a conclusion sentence and ■.

3.3 Exercises

Write each solution in full, using the templates.

E1. For each sequence, draw a graph (no loops, no multiple edges) with these degrees, or explain why none exists: (a) 3, 3, 2, 2, 1; (b) 4, 4, 3, 1; (c) 3, 3, 3, 3, 2, 2.

E2. A graph has 21 edges, 3 vertices of degree 4 and all other vertices of degree 3. How many vertices does it have?

E3. Prove by induction that the hypercube Q_n has $n2^{n-1}$ edges. (Hint for the step: Q_{n+1} is two copies of Q_n.) Then check the formula with the Handshaking Theorem.

E4. In Q_3, how many paths of length 3 go from 000 to 111? List them.

E5. For which $m, n \geq 1$ does $K_{m,n}$ have an Eulerian tour? An Eulerian trail that is not a tour?

E6. For which $n \geq 1$ is Q_n Eulerian? Justify with Euler's theorem.

E7. Prove that if every vertex of a (finite) graph has degree at least 2, then the graph contains a cycle.

E8. Let T be a connected graph with $n \geq 2$ vertices and exactly $n - 1$ edges. Prove that T has at least two vertices of degree 1.

E9. Is Q_4 planar? (Use Section 1.6 and Section 1.7.)

E10. (a) Explain why there is no 3-regular graph on 7 vertices. (b) Give a 3-regular graph on 6 vertices with no triangle. (c) Give a 4-regular graph on 8 vertices with no triangle, and explain why this does not contradict Problem 10.

A Answers to the "spot the error" puzzles

Spot the error 1 (one tail per arc). An arc has two ends, but only one of them is its *initial* vertex. So each arc contributes exactly 1 to $\sum \deg^+$ and 1 to $\sum \deg^-$: $\sum \deg^+(v) = \sum \deg^-(v) = |E|$. The "two ends" give $\sum \big(\deg^-(v) + \deg^+(v)\big) = 2|E|$.

Spot the error 2 (the new vertex). "Each degree goes up by 1" is true only for the k old vertices. The new vertex was not in K_k, so the hypothesis says nothing about it; its degree must be computed separately: it is joined to the k old vertices, so it also has degree k. With that sentence added, the proof is complete (compare Problem 3).

Spot the error 3 (connected). Euler's theorem assumes G is connected. A walk cannot move from one triangle to the other, so no walk uses all six edges. Each triangle on its own is Eulerian.

Spot the error 4 (the smaller graph may be disconnected). The inductive hypothesis is about *connected* graphs on n vertices, but removing an arbitrary vertex can disconnect the graph: removing the centre of $K_{1,5}$ leaves 5 isolated vertices, and the hypothesis cannot be applied. (The conclusion happens to be true, but the argument does not prove it.) Fix: remove a vertex of degree 1, and handle the case "all degrees ≥ 2" by handshaking, as in Problem 8.

Spot the error 5 (a bound that holds proves nothing). $e \leq 3v - 6$ is a *necessary* condition for planarity, not a sufficient one: satisfying it does not make a graph planar. For $K_{3,3}$ the right bound is the triangle-free one, $e \leq 2v - 4 = 8$, which fails since $e = 9$. So $K_{3,3}$ is *not* planar.

Spot the error 6 (no contradiction was found). Mantel's theorem says a triangle-free graph on 9 vertices has *at most* 20 edges. Our graph has $18 \leq 20$, which is perfectly consistent, so this gives no contradiction. "Very dense, so impossible" is not an argument. A correct proof must use the structure (all degrees exactly 4), as in Problem 10.