

Tutorial 6: Assignment 2, Questions 8 to 14

Generating functions with weights, partitions, exponential generating functions, recurrences and combinatorial proofs

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Write the generating function of an equation with **coefficients**, such as $2w + 3x + 5y + 7z = n$, and handle **lower bounds** by pulling out a power of x .
2. Write the generating function for **partitions** of n with restrictions on how often a part may appear and on how large a part may be.
3. **Read off a sequence** from an exponential generating function, including ones that mix e^{ax} , polynomials and $\frac{1}{1-x}$.
4. Build the **exponential generating function** for **arrangements** of letters taken from a word with repeated letters, and explain why the $k!$ in the denominators is there.
5. **Solve a recurrence** with generating functions: multiply, sum, solve for $f(x)$, split into partial fractions, read the coefficient.
6. Write a **combinatorial proof** (count one set in two ways) and an **algebraic proof** of a binomial identity.

Sections 1.1 to 1.6 review the tools with a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 solves Questions 8 to 14 of Assignment 2 in full, with the alternative methods that appeared in student solutions. Section 3 collects the classic mistakes, “spot the error” puzzles, a checklist and extra exercises. Tutorials 1 to 5 (counting rules, $\binom{n}{r}$, stars and bars, induction, inclusion and exclusion, generating functions for distributions) are assumed.

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1 The toolbox

The second half of Assignment 2 is the generating function chapter. Questions 8 and 9 build **ordinary** generating functions for equations with coefficients and for partitions. Questions 10 and 11 introduce **exponential** generating functions, the right tool when *order* matters. Question 12 uses generating functions as a machine for **solving recurrences**. Questions 13 and 14 return to binomial coefficients with **combinatorial proofs**. As always, each tool comes with the idea, a small example, the precise statement, a bigger example, and a recap. The examples in this section are *not* the assignment questions; those are solved in Section 2.

1.1 Equations with coefficients: one factor per variable

The intuition

In Tutorial 5 every variable contributed 1 unit per step: the factor was $1 + x + x^2 + \dots$. In $2w + 3x + 5y + 7z = n$ the variable w contributes 2 **units per step**: $2w$ can be 0, 2, 4, 6, ... So its factor is $1 + t^2 + t^4 + \dots$, which is the geometric series in t^2 :

$$1 + t^2 + t^4 + \dots = \frac{1}{1 - t^2}.$$

Same rule as before: **one factor per variable, exponents = what that variable can contribute to the total**. A lower bound $w \geq 3$ means $2w \geq 6$, so the factor starts at t^6 : $t^6 + t^8 + \dots = \frac{t^6}{1 - t^2}$.

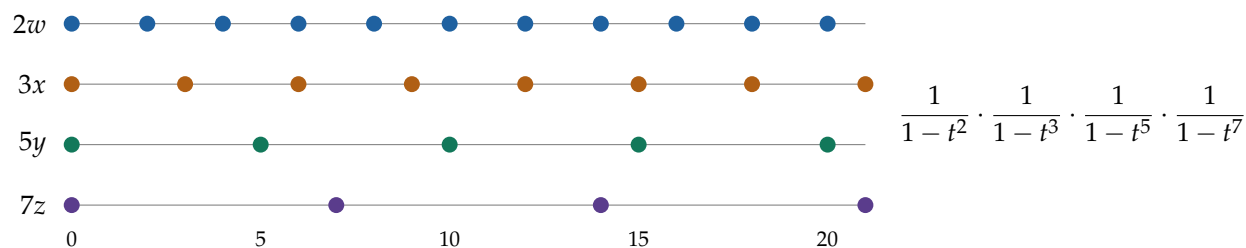


Figure 1. The possible contributions of each term of $2w + 3x + 5y + 7z$ to the total. Each row is one factor: its dots are the exponents that appear in it.

Simple example: making change

In how many ways can you make 15 cents with pennies, nickels and dimes? Let p, q, d be the numbers of coins: $p + 5q + 10d = 15$. The generating function is $\frac{1}{(1-t)(1-t^5)(1-t^{10})}$, and the answer is its coefficient of t^{15} . By listing: $d = 1$: $q \in \{0, 1\}$ (2 ways); $d = 0$: $q \in \{0, 1, 2, 3\}$ (4 ways). Total 6. ✓

Generating functions with coefficients

Let $a_1, \dots, a_k$ be positive integers and $m_1, \dots, m_k \geq 0$. The number of integer solutions of

$$a_1 y_1 + a_2 y_2 + \dots + a_k y_k = n, \quad y_i \geq m_i,$$

is the coefficient of t^n in

$$f(t) = \prod_{i=1}^k (t^{a_i m_i} + t^{a_i(m_i+1)} + t^{a_i(m_i+2)} + \dots) = \prod_{i=1}^k \frac{t^{a_i m_i}}{1 - t^{a_i}} = \frac{t^{a_1 m_1 + \dots + a_k m_k}}{\prod_i (1 - t^{a_i})}.$$

The power of t in front is the **smallest possible total**. Use a letter for the generating function variable that is not one of the unknowns.

Comprehensive example: $3a + 4b + c = n$ with $a \geq 2, b \geq 1$

Solution. $3a \in \{6, 9, 12, \dots\}$ gives $\frac{t^6}{1-t^3}$; $4b \in \{4, 8, \dots\}$ gives $\frac{t^4}{1-t^4}$; $c \in \{0, 1, \dots\}$ gives $\frac{1}{1-t}$. So

$$f(t) = \frac{t^{10}}{(1-t)(1-t^3)(1-t^4)}.$$

Check. The smallest total is $3 \cdot 2 + 4 \cdot 1 + 0 = 10$, the exponent in front. For $n = 11$: $(a, b, c) = (2, 1, 1)$ only, and $[x^{11}]f = [x^1] \frac{1}{(1-t)(1-t^3)(1-t^4)} = 1$. ✓

Two slips with weighted variables

- **Lower bound on the variable, not on the term.** $x \geq 4$ in $3x$ means the term is at least 12: factor $\frac{t^{12}}{1-t^3}$, not $\frac{t^4}{1-t^3}$.
- **The same letter twice.** In $2w + 3x + 5y + 7z$, the letter x is an unknown. Writing the generating function in x as well makes " x^{12} " ambiguous. Use t (or a , as one student did).

Weighted variables in one breath

- Term $a y$ with $y \geq m$: factor $\frac{t^{am}}{1-t^a}$.
- Multiply the factors; the answer for total n is $[x^n]$ of the product.

1.2 Partitions of an integer

The intuition

A **partition** of n writes n as a sum of positive integers where **order does not matter**: $3 + 1 + 1$ and $1 + 3 + 1$ are the same partition. (Compare compositions in Tutorial 5, where order matters.) Since order does not matter, a partition is described by **how many times each part is used**: y_1 ones, y_2 twos, y_3 threes, $\dots$, with

$$1 \cdot y_1 + 2 \cdot y_2 + 3 \cdot y_3 + \dots = n.$$

That is an equation with coefficients, so Section 1.1 applies: the part i contributes the factor $1 + x^i + x^{2i} + \dots$. Restrictions are restrictions on the y_i : “part i at most 5 times” cuts the factor after x^{5i} , and “no part larger than 12” removes the factors for $i \geq 13$.

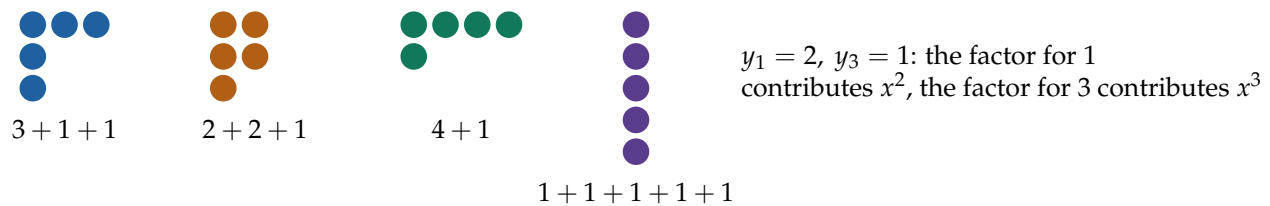


Figure 2. Some partitions of 5 drawn as dot diagrams (rows in decreasing order). Order does not matter, so a partition is just the list of how many times each part is used.

Simple example: the partitions of 5

5, $4 + 1$, $3 + 2$, $3 + 1 + 1$, $2 + 2 + 1$, $2 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1$: there are 7. The generating function $\prod_{i \geq 1} \frac{1}{1 - x^i}$ starts $1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \dots$, and its coefficient of x^5 is 7. ✓

Generating functions for partitions

Let $p(n)$ be the number of partitions of n (with $p(0) = 1$). Then

$$\sum_{n \geq 0} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}.$$

Restrictions. Restrictions on the number y_i of copies of part i change the i -th factor:

Restriction	Factor for part i
no restriction	$1 + x^i + x^{2i} + \dots = \frac{1}{1 - x^i}$
part i used at most k times	$1 + x^i + \dots + x^{ki} = \frac{1 - x^{(k+1)i}}{1 - x^i}$
distinct parts (at most once)	$1 + x^i$
part i not allowed	1 (drop the factor)

“All parts at most m ” means: parts $i > m$ are not allowed, so the product stops at $i = m$.

Comprehensive example: distinct parts versus odd parts

Show that the number of partitions of n into **distinct** parts equals the number into **odd** parts.

Solution. Distinct parts: $\prod_{i \geq 1} (1 + x^i)$. Odd parts: $\prod_{j \geq 1} \frac{1}{1 - x^{2j-1}}$. Now

$$\prod_{i \geq 1} (1 + x^i) = \prod_{i \geq 1} \frac{1 - x^{2i}}{1 - x^i} = \frac{(1 - x^2)(1 - x^4)(1 - x^6) \cdots}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \cdots} = \frac{1}{(1 - x)(1 - x^3)(1 - x^5) \cdots},$$

since every even factor in the denominator cancels with the numerator. Equal generating functions mean equal coefficients. Check ($n = 6$): distinct: $6, 5 + 1, 4 + 2, 3 + 2 + 1$; odd: $5 + 1, 3 + 3, 3 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1$. Four each. ✓

“At most 5 times” is about each part, not about the number of parts

“Summands cannot occur more than five times” restricts how often *each* value repeats: $2 + 2 + 2 + 2 + 2 + 1$ is fine (2 appears five times), but $1 + 1 + 1 + 1 + 1 + 1$ is not. It does *not* say “at most five summands”. That would be a different generating function.

Partitions in one breath

- One factor per part size i ; its exponents are the multiples of i allowed.
- At most k copies: $\frac{1 - x^{(k+1)i}}{1 - x^i}$; parts at most m : product up to $i = m$.

1.3 Exponential generating functions: reading off a sequence**The intuition**

An ordinary generating function stores a_n as the coefficient of x^n . An **exponential** generating function stores a_n as the coefficient of $\frac{x^n}{n!}$. It is the same idea with a different “filing system”, and it is chosen because the most important function in it is very simple:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{is the EGF of } 1, 1, 1, 1, \dots$$

To read a_n from an EGF $f(x)$, find the coefficient of x^n and **multiply by $n!$** : $a_n = n! \cdot [x^n]f(x)$.

EGF $f(x)$	coefficient of x^n	sequence $a_n = n! \cdot [x^n]f$
e^x	$\frac{1}{n!}$	$1, 1, 1, 1, \dots$
e^{ax}	$\frac{a^n}{n!}$	$1, a, a^2, a^3, \dots$
$c x^k$ (one term)	c at $n = k$, else 0	$c \cdot k!$ at position k , 0 elsewhere
$\frac{1}{1-x}$	1	$n!$: $1, 1, 2, 6, 24, \dots$
$\frac{1}{1-ax}$	a^n	$a^n n!$

Table 1. The dictionary for Question 10. The last two rows are the surprising ones: $\frac{1}{1-x}$ is “simple” as an ordinary generating function, but as an EGF it stores $n!$.

Simple example: $f(x) = e^{2x} + x$

e^{2x} gives 2^n . The single term $x = 1 \cdot \frac{x^1}{1!}$ adds 1 at position 1. So the sequence is 1, $2 + 1 = 3$, 4, 8, 16, ..., that is $a_1 = 3$ and $a_n = 2^n$ for $n \neq 1$.

Exponential generating function

The **exponential generating function** (EGF) of $a_0, a_1, a_2, \dots$ is

$$f(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots$$

Linearity. If f is the EGF of (a_n) and g of (b_n) , then $\alpha f + \beta g$ is the EGF of $(\alpha a_n + \beta b_n)$. So split f into pieces from Table 1, read each piece, and add position by position. **Polynomials.** A term $c x^k$ equals $c k! \cdot \frac{x^k}{k!}$, so it adds $c k!$ to a_k only.

Comprehensive example: $f(x) = 2e^{-x} + 4x^3 + \frac{1}{1-x}$

n	0	1	2	3	4	general n
$2e^{-x}$	2	-2	2	-2	2	$2(-1)^n$
$4x^3$	0	0	0	$4 \cdot 3! = 24$	0	
$\frac{1}{1-x}$	1	1	2	6	24	$n!$
a_n	3	-1	4	28	26	$2(-1)^n + n!$ for $n \neq 3$

Forgetting the $n!$

The coefficient of x^n is *not* a_n . For $e^x + x^2$ the coefficient of x^2 is $\frac{1}{2} + 1 = \frac{3}{2}$, and $a_2 = 2! \cdot \frac{3}{2} = 3$, not $\frac{3}{2}$ and not 2. Always rewrite each term as (something) $\cdot \frac{x^n}{n!}$ before reading.

Reading an EGF in one breath

- $a_n = n! \cdot [x^n]f$. Split f into pieces, read each with Table 1, add.
- $e^{ax} \mapsto a^n$; $cx^k \mapsto c k!$ at $n = k$; $\frac{1}{1-x} \mapsto n!$.

1.4 Exponential generating functions for arrangements

The intuition

To count **selections** of letters (order does not matter), an ordinary generating function works: the letter A that may be used 0, 1 or 2 times contributes $1 + x + x^2$. To count **arrangements** (order matters), we need the number of words, and a word that uses A twice, B once and C once has $\frac{4!}{2!1!1!}$ arrangements. The $\frac{1}{2!}$ is about the A's alone, so it can be built into A's factor: $1 + x + \frac{x^2}{2!}$. When the factors are multiplied, the denominators multiply to $2!1!1!$, and reading the coefficient of $\frac{x^4}{4!}$ supplies the $4!$ on top. That is exactly the EGF.

pick $\frac{x^2}{2!} \cdot x \cdot x = \frac{x^4}{2!1!1!}$
 $= \frac{4!}{2!1!1!} \cdot \frac{x^4}{4!}$: 12 words
 using A, A, B, C

Figure 3. Why EGFs count arrangements. Each choice of one term per letter is a choice of how many copies of each letter to use, and it contributes the number of distinct arrangements of those copies.

Simple example: two letters from AAB

Words of length 2 using letters of AAB: AA, AB, BA. That is 3.

EGF: $(1 + x + \frac{x^2}{2!})(1 + x) = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3$. Coefficient of x^2 is $\frac{3}{2}$, times $2!$: 3. ✓
 (The ordinary GF $(1 + x + x^2)(1 + x)$ gives 2 at x^2 : the selections $\{A, A\}$, $\{A, B\}$.)

EGF for arrangements

Suppose there are r kinds of objects and objects of kind j may be used k_j times for k_j in an allowed set K_j . The number of arrangements (words) of length n is $n! \cdot [x^n]$ of

$$f(x) = \prod_{j=1}^r \left(\sum_{k \in K_j} \frac{x^k}{k!} \right).$$

Reason. A choice of exponents $k_1 + \dots + k_r = n$ contributes $\frac{x^n}{k_1! \dots k_r!} = \binom{n}{k_1, \dots, k_r} \frac{x^n}{n!}$, and $\binom{n}{k_1, \dots, k_r} = \frac{n!}{k_1! \dots k_r!}$ is the number of arrangements with exactly those multiplicities.

Typical factors: at most m copies: $1 + x + \dots + \frac{x^m}{m!}$; at least 2 copies: $\frac{x^2}{2!} + \frac{x^3}{3!} + \dots$; any number: e^x ; an even number: $\frac{e^x + e^{-x}}{2}$.

Comprehensive example: three letters from BANANA

BANANA has B once, A three times, N twice. So

$$f(x) = (1 + x) \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \right) \left(1 + x + \frac{x^2}{2!} \right).$$

The terms giving x^3 , with their multiplicities (B, A, N) : $(1, 2, 0)$: $\frac{1}{2}$; $(1, 0, 2)$: $\frac{1}{2}$; $(1, 1, 1)$: 1; $(0, 3, 0)$: $\frac{1}{6}$; $(0, 2, 1)$: $\frac{1}{2}$; $(0, 1, 2)$: $\frac{1}{2}$. Sum $\frac{19}{6}$, so there are $3! \cdot \frac{19}{6} = 19$ words of length 3.

Check by complement. All $3^3 = 27$ words over $\{B, A, N\}$, minus those with ≥ 2 B's ($3 \cdot 2 + 1 = 7$), minus NNN (1): $27 - 8 = 19$. ✓

Selections versus arrangements

"In how many ways can we *arrange* n letters selected from ..." asks for words: EGF. "In how many ways can we *select* ..." asks for multisets: ordinary GF. Using $1 + x + x^2$ for a letter in an arrangement question is the most common mistake in Question 11.

EGFs for arrangements in one breath

- Letter allowed $0, \dots, m$ times: $1 + x + \frac{x^2}{2!} + \dots + \frac{x^m}{m!}$. Multiply over the letter types.

- Number of words of length n : $n! \cdot [x^n]f$. Check: the full length gives $\frac{(\text{length})!}{\prod(\text{multiplicities})!}$.

1.5 Solving recurrences with generating functions

The intuition

A recurrence is infinitely many equations, one for each n . A generating function packs the whole sequence into one function $f(x) = \sum a_n x^n$. So: **multiply** the n -th equation by a power of x , **add** all of them, and you get *one* equation for $f(x)$. Solve it with ordinary algebra, then **unpack** $f(x)$ with partial fractions and the series you know. The initial values enter when you rewrite the shifted sums in terms of $f(x)$.

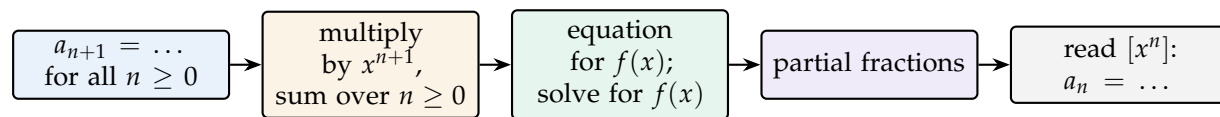


Figure 4. The five steps of the method. Multiply by the **highest** power that appears (x^{n+1} for a first-order recurrence, x^{n+2} for a second-order one), so every sum starts cleanly.

Simple example: $a_{n+1} = 2a_n$, $a_0 = 3$

Multiply by x^{n+1} and sum over $n \geq 0$: $\sum_{n \geq 0} a_{n+1} x^{n+1} = 2x \sum_{n \geq 0} a_n x^n$. The left sum is $f(x)$ without its first term: $f(x) - a_0 = 2xf(x)$. So $f(x)(1 - 2x) = 3$, $f(x) = \frac{3}{1-2x} = \sum 3 \cdot 2^n x^n$, and $a_n = 3 \cdot 2^n$. ✓

The shifted sums and the series you need

With $f(x) = \sum_{n \geq 0} a_n x^n$:

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = f(x) - a_0, \quad \sum_{n \geq 0} a_{n+2} x^{n+2} = f(x) - a_0 - a_1 x, \quad \sum_{n \geq 0} a_n x^{n+k} = x^k f(x).$$

Series to recognise.

function	coefficient of x^n	function	coefficient of x^n
$\frac{1}{1-ax}$	a^n	$\frac{x}{(1-x)^2}$	n
$\frac{1}{(1-x)^k}$	$\binom{n+k-1}{k-1}$	$\frac{x(1+x)}{(1-x)^3}$	n^2

The last two come from $\frac{1}{1-x} = \sum x^n$ by applying “differentiate, then multiply by x ” once and twice. **Partial fractions.** For distinct $a \neq b$: $\frac{p(x)}{(1-ax)(1-bx)} = \frac{A}{1-ax} + \frac{B}{1-bx}$; find A by putting $x = \frac{1}{a}$ in $p(x) = A(1-bx) + B(1-ax)$, and B with $x = \frac{1}{b}$. A repeated factor $(1-x)^k$ needs the terms $\frac{A_1}{1-x} + \dots + \frac{A_k}{(1-x)^k}$.

Comprehensive example: $a_{n+2} = a_{n+1} + 2a_n$, $a_0 = 0$, $a_1 = 3$

Solution. Multiply by x^{n+2} and sum over $n \geq 0$:

$$(f(x) - a_0 - a_1x) = x(f(x) - a_0) + 2x^2f(x),$$

so $f(x)(1 - x - 2x^2) = 3x$ and $f(x) = \frac{3x}{(1-2x)(1+x)}$. Partial fractions: $3x = A(1+x) + B(1-2x)$. With $x = \frac{1}{2}$: $\frac{3}{2} = \frac{3}{2}A$, so $A = 1$. With $x = -1$: $-3 = 3B$, so $B = -1$. Hence $f(x) = \frac{1}{1-2x} - \frac{1}{1+x}$ and $a_n = 2^n - (-1)^n$.

Check. $a_0 = 0$, $a_1 = 3$, $a_2 = 3 = a_1 + 2a_0$, $a_3 = 9 = a_2 + 2a_1$. ✓

Two slips in the method

- **Dropping the initial values.** $\sum_{n \geq 0} a_{n+1}x^{n+1}$ is $f(x) - a_0$, not $f(x)$. Without $-a_0$ the answer cannot depend on a_0 , which is clearly wrong.
- **Shifting the right-hand side wrongly.** $\sum_{n \geq 0} 3^n x^{n+1} = x \sum_{n \geq 0} (3x)^n = \frac{x}{1-3x}$: the extra x comes from multiplying by x^{n+1} , not x^n .

Recurrences by generating functions in one breath

- Multiply by $x^{n+(\text{order})}$, sum, replace shifted sums by $f(x)$ minus initial terms, solve for $f(x)$.
- Partial fractions, then read a_n from known series. Check a_0, a_1, a_2 against the recurrence.

1.6 Combinatorial proofs

The intuition

To prove LHS = RHS for two counting expressions, find **one set** and count it in **two different ways**: one way gives the LHS, the other gives the RHS. Both count the same set, so they are equal. No algebra is needed, and the proof often explains *why* the identity is true.

The classic set for identities with a factor k or n in front: **committees with a chair**. Choosing “a committee, then its chair” gives a factor k (the committee size); choosing “the chair, then the rest of the committee” gives a factor n (the class size).

Way 1: committee first



$\binom{n}{k}$ committees,
then k choices of chair

Way 2: chair first



n choices of chair,
then $\binom{n-1}{k-1}$ other members

Figure 5. The same object, a committee of $k = 3$ from $n = 6$ people with an orange chair, built in two orders. Both orders produce every (committee, chair) pair exactly once.

Simple example: $\binom{n}{k} = \binom{n}{n-k}$

Count the k -person teams from n people. Way 1: choose the team, $\binom{n}{k}$. Way 2: choose the $n - k$ people who are *left out*, $\binom{n}{n-k}$; the team is everyone else. Both count teams, so they are equal.

How to write a combinatorial proof

- (i) **Define the set.** Describe precisely which objects are counted (“pairs (C, c) where C is a k -subset of $\{1, \dots, n\}$ and $c \in C$ ”).
- (ii) **Count it one way.** Justify each factor with the rule of product or sum; this gives the LHS.
- (iii) **Count it another way.** Again justify each factor; this gives the RHS.
- (iv) **Conclude.** “Since both expressions count the same set, they are equal.”

An **algebraic proof** instead rewrites one side into the other using definitions such as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Both are valid proofs; a question that asks for one of them wants that one.

Comprehensive example: a chair and a secretary

Prove $\sum_{k=2}^n k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$ for $n \geq 2$.

Solution. Count the committees (of any size) from n people with a chair and a different secretary.
Way 1. For a committee of size k : $\binom{n}{k}$ committees, k chairs, $k - 1$ secretaries. Adding over k (rule of sum; sizes 0, 1 contribute 0) gives the LHS. *Way 2.* Choose the chair (n ways), then the secretary ($n - 1$ ways), then for each of the other $n - 2$ people decide in or out (2^{n-2} ways). This gives the RHS. ■

Check ($n = 3$): $2 \cdot 1 \cdot 3 + 3 \cdot 2 \cdot 1 = 12 = 3 \cdot 2 \cdot 2$. ✓

The two counts must count the same set

A common slip is to describe two procedures that count *different* things, for example “choose a committee and its *youngest* member” on one side (only one choice per committee, so the factor k disappears) and “choose a chair and the rest” on the other. Name the set once, at the start, and check that each procedure produces every element exactly once.

Combinatorial proofs in one breath

- One set, two counts. Name the set first.
- Factor k in front: “pick a special member of the chosen group”; factor n : “pick the special member first”.

The toolbox at a glance

Situation	Count / method	Signal words
Equation with coefficients	factor $\frac{t^m}{1-t^m}$ per term $ay, y \geq m$	" $2w + 3x + 5y + 7z = n$ "
Partitions with restrictions	factor per part size i	"partitions of n into summands"
Reading an EGF	$a_n = n![x^n]f$; Table 1	"sequence generated by the exponential..."
Arrangements of letters	$\prod (1 + x + \dots + \frac{x^m}{m!})$	"arrange n letters selected from"
Recurrence	multiply, sum, solve, split, read	"by the method of generating functions"
Identity with binomials	one set, two counts; or factorials	"combinatorial proof", "algebraic proof"

2 The assignment problems, worked in full

2.1 How to write the solutions

Questions 8 to 11 ask for generating functions or sequences: give the function (or sequence), and say *why* each factor looks the way it does. Question 12 asks to *solve* recurrences: every step of the method must be visible. Questions 13 and 14 are proofs. As before, the assignment asks you to **show all the steps**.

Writing template for a generating function answer

Statement. [What is being counted, and which coefficient of the generating function will give it.]

Solution. [One line per variable / part / letter: what it may contribute, and the factor that records this. Then the product, and a simplified closed form.]

Answer. [The generating function, boxed, and which coefficient to read.]

Check (optional but recommended). [A small coefficient computed by hand, or the full-length count.]

Writing template for solving a recurrence with generating functions

1. Set-up. ["Let $f(x) = \sum_{n \geq 0} a_n x^n$."]

2. Multiply and sum. ["Multiplying the recurrence by x^{n+k} and summing over $n \geq 0, \dots$ "]

3. Rewrite with $f(x)$. [Replace each shifted sum by $f(x)$ minus initial terms; use the initial values.]

4. Solve and split. [Solve for $f(x)$; partial fractions with the constants found.]

5. Read off and check. [$a_n = \dots$; verify a_0, a_1, a_2 .]

Writing template for a combinatorial proof

Statement. [The identity and the range of n, k .]

The set. ["We count the set of $\dots$ in two ways."]

First count. [Step by step, naming the rules; this gives the LHS.]

Second count. [Step by step; this gives the RHS.]

Conclusion. ["Both count the same set, so $LHS = RHS$." ■]

Phrasebook: new sentences for this tutorial

Where	What to write
Weighted variable	"Since $x \geq 4$, the term $3x$ takes the values $12, 15, 18, \dots$, recorded by $t^{12} + t^{15} + \dots = \frac{t^{12}}{1-t^3}$."
Partitions	"Part i may be used $0, 1, \dots, 5$ times, so it contributes $1 + x^i + \dots + x^{5i} = \frac{1-x^{6i}}{1-x^i}$."
Reading an EGF	"Since $x^2 = 2 \cdot \frac{x^2}{2!}$, the term x^2 adds 2 to a_2 ."
EGF factor	"The letter I may be used 0 to 4 times; it contributes $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$."
Shifted sum	" $\sum_{n \geq 0} a_{n+2} x^{n+2} = f(x) - a_0 - a_1 x$."
Partial fractions	"Writing $\frac{1+3x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$ and putting $x = 1$ and $x = \frac{1}{2}$ gives $A = -4, B = 5$."
Combinatorial proof	"We count the committees of k students from a class of n with one member chosen as chair, in two ways."

2.2 Problem 8: generating functions for weighted equations

Problem 8 (Exercises 9.3, #4)

Find the generating function for the number of integer solutions of

(a) $2w + 3x + 5y + 7z = n$, $0 \leq w, x, y, z$; (b) $2w + 3x + 5y + 7z = n$, $0 \leq w, 4 \leq x, y, 5 \leq z$.

How to think about it. Section 1.1. Since x is one of the unknowns, use t for the generating function. In (b), " $4 \leq x, y$ " means $x \geq 4$ and $y \geq 4$.

Model solution to Problem 8

Solution (what you write)

Statement. Find $f(t)$ such that the number of solutions for each n is $[x^n]f(t)$.

(a) Each term contributes a non-negative multiple of its coefficient: $2w \in \{0, 2, 4, \dots\}$, $3x \in \{0, 3, 6, \dots\}$, $5y \in \{0, 5, 10, \dots\}$, $7z \in \{0, 7, 14, \dots\}$. So

$$\begin{aligned}
 f(t) &= (1 + t^2 + t^4 + \dots)(1 + t^3 + t^6 + \dots) \\
 &\quad \cdot (1 + t^5 + t^{10} + \dots)(1 + t^7 + t^{14} + \dots) \\
 &= \frac{1}{(1-t^2)(1-t^3)(1-t^5)(1-t^7)}.
 \end{aligned}$$

(b) Now $2w \in \{0, 2, \dots\}$; $x \geq 4$ gives $3x \in \{12, 15, \dots\}$; $y \geq 4$ gives $5y \in \{20, 25, \dots\}$; $z \geq 5$ gives $7z \in \{35, 42, \dots\}$. So

$$\begin{aligned}
 f(t) &= \frac{1}{1-t^2} \cdot \frac{t^{12}}{1-t^3} \cdot \frac{t^{20}}{1-t^5} \cdot \frac{t^{35}}{1-t^7} \\
 &= \frac{t^{67}}{(1-t^2)(1-t^3)(1-t^5)(1-t^7)}.
 \end{aligned}$$

Commentary (what it does)

The answer is a function of t that works for every n at once.

One factor per variable (Figure 1). The product is the number of ways to choose one contribution from each term with the right total.

Lower bound on the variable: multiply by its coefficient. $12 + 20 + 35 = 67$.

Check. In (a), $[x^5]f = 2$: $(w, x, y, z) = (1, 1, 0, 0)$ and $(0, 0, 1, 0)$. In (b), the smallest n with a solution is 67, with the single solution $(0, 4, 4, 5)$, and indeed $[x^{67}]f = 1$. ✓

Expanding (a): $1 + t^2 + t^3 + t^4 + 2t^5 + 2t^6 + 3t^7 + \dots$ (checked by computer).

2.3 Problem 9: partitions with restricted repetition

Problem 9 (Exercises 9.3, #6)

What is the generating function for the number of partitions of $n \in \mathbb{N}$ into summands that (a) cannot occur more than five times; and (b) cannot exceed 12 and cannot occur more than five times?

How to think about it. Section 1.2. “Cannot occur more than five times” cuts each factor after five copies; “cannot exceed 12” stops the product at $i = 12$.

Model solution to Problem 9

Solution (what you write)

Set-up. A partition of n is determined by the numbers $y_i \geq 0$ of summands equal to i , with $\sum_i i y_i = n$. The summand i contributes $i y_i$ to n .

(a) Each summand i is used $0, 1, \dots, 5$ times, so it contributes $0, i, 2i, \dots, 5i$, recorded by

$$1 + x^i + x^{2i} + x^{3i} + x^{4i} + x^{5i} = \frac{1 - x^{6i}}{1 - x^i}.$$

Any $i \geq 1$ may appear, so

$$f(x) = \prod_{i=1}^{\infty} (1 + x^i + \dots + x^{5i}) = \prod_{i=1}^{\infty} \frac{1 - x^{6i}}{1 - x^i}.$$

(b) Now only $i = 1, \dots, 12$ may appear (the factor for $i \geq 13$ is 1), so

$$f(x) = \prod_{i=1}^{12} (1 + x^i + \dots + x^{5i}) = \prod_{i=1}^{12} \frac{1 - x^{6i}}{1 - x^i}.$$

Check. (a) starts $1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 10x^6 + \dots$. Compared with $p(6) = 11$, the one partition lost is $1 + 1 + 1 + 1 + 1 + 1$ (six 1s). ✓

Commentary (what it does)

This turns partitions into Section 1.1.

Finite geometric sum with ratio x^i :
 $\frac{1 - (x^i)^6}{1 - x^i}$.

For a fixed n only the factors with $i \leq n$ matter, so the infinite product is fine.

“Cannot exceed 12” is about the size of a summand.

Coefficients confirmed by computer up to $n = 25$.

A beautiful simplification (Glaisher)

In $\prod_{i \geq 1} \frac{1 - x^{6i}}{1 - x^i}$, every numerator factor $1 - x^{6i}$ cancels the denominator factor $1 - x^j$ with $j = 6i$. What is left is $\prod_{6 \nmid j} \frac{1}{1 - x^j}$: the generating function for partitions with **no part divisible by 6**. So “no part used more than 5 times” and “no part divisible by 6” are equally common, for every n .

This is the same trick as the distinct-parts example of Section 1.2 (there: 2 instead of 6).

2.4 Problem 10: sequences from exponential generating functions

Problem 10 (Exercises 9.4, #2)

Determine the sequence generated by each of the following exponential generating functions.

(a) $f(x) = 3e^{3x}$ (b) $f(x) = 6e^{5x} - 3e^{2x}$ (c) $f(x) = e^x + x^2$ (d) $f(x) = e^{2x} - 3x^3 + 5x^2 + 7x$

(e) $f(x) = \frac{1}{1-x}$ (f) $f(x) = \frac{3}{1-2x} + e^x$

How to think about it. Section 1.3: split into pieces from Table 1, read each piece, add position by position.

Model solution to Problem 10

Solution (what you write)

(a) $3e^{3x} = 3 \sum_{n \geq 0} 3^n \frac{x^n}{n!} = \sum_{n \geq 0} 3^{n+1} \frac{x^n}{n!}$. So $a_n = \boxed{3^{n+1}}$:
3, 9, 27, 81, ...

(b) By linearity, $a_n = \boxed{6 \cdot 5^n - 3 \cdot 2^n}$: 3, 24, 138, 726, ...

(c) e^x gives 1 for every n . The term $x^2 = 2 \cdot \frac{x^2}{2!}$ adds 2 at $n = 2$.
So $a_2 = 1 + 2 = 3$ and $a_n = 1$ otherwise: $\boxed{1, 1, 3, 1, 1, \dots}$

(d) e^{2x} gives 2^n . The polynomial adds $7 \cdot 1! = 7$ at $n = 1$,
 $5 \cdot 2! = 10$ at $n = 2$, and $-3 \cdot 3! = -18$ at $n = 3$:

n	0	1	2	3	$n \geq 4$
2^n	1	2	4	8	2^n
polynomial	0	7	10	-18	0
a_n	1	9	14	-10	2^n

So the sequence is $\boxed{1, 9, 14, -10, 16, 32, 64, \dots}$, with $a_n = 2^n$ for $n \geq 4$.

(e) $\frac{1}{1-x} = \sum_{n \geq 0} x^n = \sum_{n \geq 0} n! \frac{x^n}{n!}$, so $a_n = \boxed{n!}$: 1, 1, 2, 6, 24, ...

(f) $\frac{3}{1-2x} = \sum 3 \cdot 2^n x^n = \sum 3 \cdot 2^n n! \frac{x^n}{n!}$, and e^x adds 1. So $a_n =$
 $\boxed{3 \cdot 2^n \cdot n! + 1}$: 4, 7, 25, 145, 1153, ...

Commentary (what it does)

$e^{ax} \mapsto a^n$, then multiply by 3.

$n = 2$: $150 - 12 = 138$.

Not 1 + 1: the polynomial term gets multiplied by 2!

The table keeps the bookkeeping honest.

Rewrite $x^n = n! \cdot \frac{x^n}{n!}$ inside the sum, term by term.

$n = 3$: $3 \cdot 3 \cdot 8 \cdot 6 + 1 = 145$.

2.5 Problem 11: arrangements of letters

Problem 11 (Exercises 9.4, #6)

- (a) Find the exponential generating function for the number of ways to arrange n letters, $n \geq 0$, selected from each of the following words. (i) HAWAII (ii) MISSISSIPPI (iii) ISOMORPHISM
 (b) For section (ii) of part (a), what is the exponential generating function if the arrangement must contain at least two I's?

How to think about it. Section 1.4. Count the letters of each word first; each letter type gives one factor that stops at its multiplicity.

Model solution to Problem 11

Solution (what you write)

Set-up. If a letter appears m times in the word, an arrangement may use it $0, 1, \dots, m$ times, recorded by $1 + x + \frac{x^2}{2!} + \dots + \frac{x^m}{m!}$. The number of arrangements of n letters is $n! \cdot [x^n]$ of the product over the letter types.

(a)(i) **HAWAII:** H 1, A 2, W 1, I 2.

$$f(x) = \left(1 + x\right)^2 \left(1 + x + \frac{x^2}{2!}\right)^2.$$

(a)(ii) **MISSISSIPPI:** M 1, I 4, S 4, P 2.

$$f(x) = \left(1 + x\right) \left(1 + x + \frac{x^2}{2!}\right) \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)^2.$$

(a)(iii) **ISOMORPHISM:** I 2, S 2, O 2, M 2, R 1, P 1, H 1.

$$f(x) = \left(1 + x\right)^3 \left(1 + x + \frac{x^2}{2!}\right)^4.$$

(b) At least two I's: the I factor keeps only $\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$, the others are unchanged:

$$f(x) = \left(1 + x\right) \left(1 + x + \frac{x^2}{2!}\right) \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) \cdot \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right).$$

Commentary (what it does)

Section 1.4: the $\frac{1}{k!}$ removes the orderings of identical copies.

Full length: $6! [x^6]f = 6! \cdot \frac{1}{2!2!} = 180 = \frac{6!}{2!2!} \cdot \checkmark$

Full length: $\frac{11!}{4!4!2!} = 34\,650 \cdot \checkmark$

Count carefully: I-S-O-M-O-R-P-H-I-S-M has 11 letters. Full length: $\frac{11!}{(2!)^4} = 2\,494\,800 \cdot \checkmark$

The last factor is for S. For $n = 3$ this gives 10 arrangements: IIM, IIS, IIP (three orders each) and III.

Small counts as a sanity check

For MISSISSIPPI, $n! [x^n]f$ for $n = 0, 1, 2, 3$ is 1, 4, 15, 53. For $n = 2$: all 16 words over $\{M, I, S, P\}$ except MM. $\checkmark$ For HAWAII, $n = 2$ gives 14: all 16 words over $\{H, A, W, I\}$ except HH and WW. $\checkmark$

2.6 Problem 12: recurrences by generating functions

Problem 12 (Exercises 10.4, #1)

Solve the following recurrence relations by the method of generating functions.

(a) $a_{n+1} - a_n = 3^n, n \geq 0, a_0 = 1$ (b) $a_{n+1} - a_n = n^2, n \geq 0, a_0 = 1$

(c) $a_{n+2} - 3a_{n+1} + 2a_n = 0, n \geq 0, a_0 = 1, a_1 = 6$ (d) $a_{n+2} - 2a_{n+1} + a_n = 2^n, n \geq 0, a_0 = 1, a_1 = 2$

How to think about it. Section 1.5 with the template. In every part let $f(x) = \sum_{n \geq 0} a_n x^n$.

Model solution to Problem 12(a)

Solution (what you write)

Multiply and sum. Multiplying by x^{n+1} and summing over $n \geq 0$:

$$\sum_{n \geq 0} a_{n+1} x^{n+1} - x \sum_{n \geq 0} a_n x^n = x \sum_{n \geq 0} (3x)^n.$$

Rewrite. $(f(x) - 1) - xf(x) = \frac{x}{1-3x}$, so

$$(1-x)f(x) = 1 + \frac{x}{1-3x} = \frac{1-2x}{1-3x},$$

$$f(x) = \frac{1-2x}{(1-x)(1-3x)}.$$

Split. $1-2x = A(1-3x) + B(1-x)$. Put $x = 1$: $-1 = -2A$, $A = \frac{1}{2}$. Put $x = \frac{1}{3}$: $\frac{1}{3} = \frac{2}{3}B$, $B = \frac{1}{2}$. So $f(x) = \frac{1/2}{1-x} + \frac{1/2}{1-3x}$.

Read off. $a_n = \frac{1+3^n}{2}$. Check: $a_0 = 1, a_1 = 2 = 1 + 3^0$, $a_2 = 5 = 2 + 3^1$. ✓

Commentary (what it does)

$$3^n x^{n+1} = x(3x)^n.$$

$a_0 = 1$ enters here.

Choose x to kill one term at a time.

Model solution to Problem 12(b)

Solution (what you write)

Multiply and sum. Multiplying by x^{n+1} and summing over $n \geq 0$: $(f(x) - 1) - xf(x) = x \sum_{n \geq 0} n^2 x^n$.

The series $\sum n^2 x^n$. From $\sum_{n \geq 0} x^n = \frac{1}{1-x}$, differentiate and multiply by x : $\sum nx^n = \frac{x}{(1-x)^2}$. Again: $\sum n^2 x^n = x \cdot \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}$.

Solve. $(1-x)f(x) = 1 + \frac{x^2(1+x)}{(1-x)^3}$, so

$$f(x) = \frac{1}{1-x} + \frac{x^2(1+x)}{(1-x)^4}.$$

Commentary (what it does)

Derive it in the solution, or cite it from the table in Section 1.5.

Split. $\frac{x^2 + x^3}{(1-x)^4} = \frac{-1}{1-x} + \frac{4}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4}$, so

$$f(x) = \frac{4}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4}.$$

Read off.

$$\begin{aligned} a_n &= 4 \binom{n+1}{1} - 5 \binom{n+2}{2} + 2 \binom{n+3}{3} \\ &= \boxed{1 + \frac{(n-1)n(2n-1)}{6}}. \end{aligned}$$

Check: $a_0 = 4 - 5 + 2 = 1$, $a_1 = 8 - 15 + 8 = 1$, $a_2 = 12 - 30 + 20 = 2$, $a_3 = 6 = 2 + 2^2$. ✓

Constants from $x^2 + x^3 = A(1-x)^3 + B(1-x)^2 + C(1-x) + D$: $x = 1$ gives $D = 2$; comparing coefficients gives $A = -1$, $B = 4$, $C = -5$.

The binomial form is a complete answer; the simplified form is optional.

A shortcut for (b): $\frac{1}{1-x}$ takes partial sums

Multiplying a generating function by $\frac{1}{1-x}$ replaces each coefficient by the sum of all coefficients up to it. From $f(x) = \frac{1}{1-x} (1 + x \sum_k k^2 x^k)$ we read $a_n = 1 + \sum_{k=0}^{n-1} k^2 = 1 + \frac{(n-1)n(2n-1)}{6}$ directly, which matches the binomial form. (It is also what the recurrence says: add $0^2, 1^2, \dots, (n-1)^2$ to a_0 .)

Model solution to Problem 12(c)

Solution (what you write)

Multiply and sum. Multiplying by x^{n+2} and summing over $n \geq 0$:

$$(f(x) - 1 - 6x) - 3x(f(x) - 1) + 2x^2 f(x) = 0.$$

Solve. $(1 - 3x + 2x^2)f(x) = 1 + 6x - 3x = 1 + 3x$, so $f(x) = \frac{1+3x}{(1-x)(1-2x)}$.

Split. $1 + 3x = A(1-2x) + B(1-x)$. Put $x = 1$: $4 = -A$, $A = -4$. Put $x = \frac{1}{2}$: $\frac{5}{2} = \frac{1}{2}B$, $B = 5$. So $f(x) = \frac{-4}{1-x} + \frac{5}{1-2x}$.

Read off. $a_n = \boxed{5 \cdot 2^n - 4}$. Check: $a_0 = 1$, $a_1 = 6$, $a_2 = 16 = 3 \cdot 6 - 2 \cdot 1$. ✓

Commentary (what it does)

Second order: multiply by x^{n+2} .

$$1 - 3x + 2x^2 = (1-x)(1-2x).$$

Model solution to Problem 12(d)

Solution (what you write)

Commentary (what it does)

Multiply and sum. Multiplying by x^{n+2} and summing over $n \geq 0$:

$$\begin{aligned}(f(x) - 1 - 2x) - 2x(f(x) - 1) + x^2 f(x) &= x^2 \sum_{n \geq 0} (2x)^n \\ &= \frac{x^2}{1 - 2x}.\end{aligned}$$

Solve. $(1 - 2x + x^2)f(x) = 1 + \frac{x^2}{1 - 2x} = \frac{1 - 2x + x^2}{1 - 2x}$. Since $1 - 2x + x^2 = (1 - x)^2$ cancels,

No partial fractions needed: the cancellation does it.

$$f(x) = \frac{1}{1 - 2x}.$$

Read off. $a_n = \boxed{2^n}$. Check: $a_2 = 2a_1 - a_0 + 2^0 = 4$, $a_3 = 2 \cdot 4 - 2 + 2 = 8$. ✓

2.7 Problem 13: $k \binom{n}{k} = n \binom{n-1}{k-1}$ two ways

Problem 13

Prove that if n and k are integers with $1 \leq k \leq n$ then $k \binom{n}{k} = n \binom{n-1}{k-1}$ (a) by using a combinatorial proof; (b) by using an algebraic proof.

How to think about it. Section 1.6 and Figure 5: committees with a chair.

Model solution to Problem 13(a)

Solution (what you write)

Commentary (what it does)

The set. Let $1 \leq k \leq n$. We count the ways to choose, from a class of n students, a committee of k students together with one member of the committee as its chair.

Name the set once; both counts refer to it.

First count. Choose the k committee members: $\binom{n}{k}$ ways. Then choose the chair among them: k ways. By the rule of product: $k \binom{n}{k}$.

Second count. Choose the chair among all n students: n ways. Then choose the other $k - 1$ members from the remaining $n - 1$ students: $\binom{n-1}{k-1}$ ways. By the rule of product: $n \binom{n-1}{k-1}$.

$k \geq 1$ is needed so the committee has a chair.

Conclusion. Both expressions count the same set, so $k \binom{n}{k} = n \binom{n-1}{k-1}$. ■

Model solution to Problem 13(b)

Solution (what you write)

Proof. For $1 \leq k \leq n$,

$$\begin{aligned} k \binom{n}{k} &= k \cdot \frac{n!}{k! (n-k)!} = \frac{n!}{(k-1)! (n-k)!} \\ &= \frac{n \cdot (n-1)!}{(k-1)! ((n-1) - (k-1))!} = n \binom{n-1}{k-1}. \quad \blacksquare \end{aligned}$$

Commentary (what it does)

Line 1: $k! = k \cdot (k-1)!$, valid since $k \geq 1$.

Line 2: $n! = n(n-1)!$ and $n - k = (n-1) - (k-1)$.

2.8 Problem 14: $\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$

Problem 14

Give a combinatorial proof that $\sum_{k=1}^n k \binom{n}{k} = n \cdot 2^{n-1}$.

How to think about it. The same set as Problem 13, but now the committee may have *any* size. The LHS adds over the sizes; the RHS picks the chair first and then decides for every other student “in or out”.

Model solution to Problem 14

Solution (what you write)

The set. Let $n \geq 1$. We count the ways to choose, from n students, a non-empty committee (of any size) together with one of its members as chair.

First count. Split by the size k of the committee, $1 \leq k \leq n$. For a fixed k there are $k \binom{n}{k}$ choices (Problem 13: committee, then chair). The sizes are disjoint cases, so by the rule of sum the total is $\sum_{k=1}^n k \binom{n}{k}$.

Second count. Choose the chair: n ways. Each of the other $n-1$ students is either on the committee or not, independently: 2^{n-1} ways. By the rule of product: $n \cdot 2^{n-1}$.

Conclusion. Both expressions count the same set, so $\sum_{k=1}^n k \binom{n}{k} = n \cdot 2^{n-1}$. $\blacksquare$

Commentary (what it does)

A committee with a chair cannot be empty.

Rule of sum over the cases $k = 1, \dots, n$.

Every subset of the other $n-1$ students, including the empty one, gives a valid committee.

Check ($n = 3$): $3 + 6 + 3 = 12 = 3 \cdot 4$. ✓

Two other ways to see it

Using Problem 13. $\sum_{k=1}^n k \binom{n}{k} = \sum_{k=1}^n n \binom{n-1}{k-1} = n \sum_{j=0}^{n-1} \binom{n-1}{j} = n 2^{n-1}$. This is an algebraic proof (not what the question asks, but a good check).

Calculus. Differentiate $(1+x)^n = \sum_k \binom{n}{k} x^k$ to get $n(1+x)^{n-1} = \sum_k k \binom{n}{k} x^{k-1}$, then put $x = 1$.

3 Mistakes, checklist and exercises

3.1 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **Lower bound on the variable treated as a bound on the term.** In Problem 8(b), $x \geq 4$ gives t^{12} , not t^4 .
2. **Using x for two things.** In Problem 8 the unknown x and the generating function variable must differ.
3. **"At most five times" read as "at most five parts".** In Problem 9 the limit is per summand value.
4. **Reading $[x^n]$ as a_n in an EGF.** Multiply by $n!$; the polynomial terms change by $k!$.
5. **Ordinary factors in an arrangement problem.** In Problem 11 the factors need $\frac{x^k}{k!}$.
6. **Miscounting the letters.** ISOMORPHISM has four doubled letters (I, S, O, M) and three single ones.
7. **Dropping initial values** when replacing shifted sums by $f(x)$ in Problem 12.
8. **Two procedures for two different sets** in a combinatorial proof (Problems 13, 14).

Spot the error 1: Problem 8(b)

"Solution." Since $x \geq 4$, $y \geq 4$ and $z \geq 5$, the generating function is $\frac{1}{1-t^2} \cdot \frac{t^4}{1-t^3} \cdot \frac{t^4}{1-t^5} \cdot \frac{t^5}{1-t^7} = \frac{t^{13}}{(1-t^2)(1-t^3)(1-t^5)(1-t^7)}$.

Where is the mistake? (Answer in Appendix A.)

Spot the error 2: Problem 9(a)

"Solution." The number of summands is at most five, so the generating function counts partitions into at most 5 parts: $\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$.

Where is the mistake? (Answer in Appendix A.) Hint: is $2 + 2 + 2 + 2 + 2 + 1$ allowed by the question?

Spot the error 3: Problem 10(c)

"Solution." e^x generates $1, 1, 1, \dots$ and x^2 adds 1 at position 2, so the sequence is $1, 1, 2, 1, 1, \dots$.

Where is the mistake? (Answer in Appendix A.)

Spot the error 4: Problem 10(e)

"Solution." $\frac{1}{1-x} = \sum_i x^i = \sum_i x^i \cdot \frac{i!}{i!} = i! \sum_i \frac{x^i}{i!}$, so the sequence is $0!, 1!, 2!, \dots$.

Where is the mistake? (Answer in Appendix A.) The final answer is right.

Spot the error 5: Problem 11(a)(i)

"Solution." H and W can be used 0 or 1 times, A and I 0, 1 or 2 times, so the generating function is $(1+x)^2(1+x+x^2)^2$.

Where is the mistake? (Answer in Appendix A.) Hint: compute the coefficient of x^6 .

Spot the error 6: Problem 12(a)

"Solution." Summing $a_{n+1}x^{n+1} - a_nx^{n+1} = 3^n x^{n+1}$ over $n \geq 0$ gives $f(x) - xf(x) = \frac{x}{1-3x}$, so $f(x) = \frac{x}{(1-x)(1-3x)}$ and $a_n = \frac{3^n - 1}{2}$.

Where is the mistake? (Answer in Appendix A.) Hint: what is a_0 according to this formula?

Spot the error 7: Problem 14

"Proof." On the left, choose a subset of size k of $\{1, \dots, n\}$ and then mark its *smallest* element. On the right, choose the marked element first (n ways) and then decide in or out for each of the other $n-1$ elements (2^{n-1}).

Where is the mistake? (Answer in Appendix A.)

3.2 Checklist before you hand in**Checklist**

- ☐ The generating function variable is **not** one of the unknowns.
- ☐ For each weighted variable, the factor starts at (coefficient) $\times$ (lower bound).
- ☐ For partitions, each factor matches the restriction on **that part size**; the product stops at the largest allowed part.
- ☐ When reading an EGF, I multiplied by $n!$ and handled each polynomial term separately.
- ☐ For arrangements I used $\frac{x^k}{k!}$, and the full-length coefficient gives $\frac{(\text{length})!}{\prod m_i!}$.
- ☐ In each recurrence, I multiplied by the right power of x and **subtracted the initial terms**.
- ☐ My partial-fraction constants were checked by plugging in a value of x .
- ☐ I verified a_0, a_1, a_2 against the recurrence.
- ☐ Each combinatorial proof names **one set** and counts it twice, with every factor justified.
- ☐ Every answer is **boxed**.

3.3 Exercises

Write each solution in full, using the templates.

- E1.** Find the generating function for the number of integer solutions of $2a + 3b + 4c = n$ with $a \geq 1$, $b \geq 0$, $0 \leq c \leq 2$.
- E2.** In how many ways can you make 25 cents with pennies, nickels, dimes and quarters? Write the generating function, then count by cases.
- E3.** Write the generating functions for partitions of n into (a) distinct parts, (b) odd parts, (c) parts of size at most 4, (d) parts each used at most twice. Find the coefficient of x^6 in (a) and (b).
- E4.** Determine the sequence generated by each EGF: (a) e^{-x} ; (b) xe^x ; (c) $\frac{e^x + e^{-x}}{2}$; (d) $5e^x - 2x^4$.
- E5.** Find the EGF for the number of arrangements of n letters selected from BANANA, and use it to count the arrangements of length 3 and of length 6.

- E6. Use an EGF to show that the number of strings of length n over $\{0, 1, 2\}$ with an *even* number of 0s is $\frac{3^n+1}{2}$.
- E7. Solve $a_{n+1} - 2a_n = 1, n \geq 0, a_0 = 0$ by generating functions.
- E8. Solve $a_{n+2} - 5a_{n+1} + 6a_n = 0, n \geq 0, a_0 = 1, a_1 = 4$ by generating functions.
- E9. Give a combinatorial proof that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$ for $0 \leq m \leq k \leq n$.
- E10. Give a combinatorial proof that $\sum_{k=2}^n k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$ (Section 1.6), then check it by differentiating $(1+x)^n$ twice.

A Answers to the “spot the error” puzzles

Spot the error 1 (bound on the variable). $x \geq 4$ means the *term* $3x \geq 12$, so its factor is $\frac{t^{12}}{1-t^3}$. In the same way $5y \geq 20$ and $7z \geq 35$. Correct: $\frac{t^{67}}{(1-t^2)(1-t^3)(1-t^5)(1-t^7)}$.

Spot the error 2 (per part, not number of parts). The question limits how often *each* summand value repeats, not the number of summands. The partition $2 + 2 + 2 + 2 + 2 + 1$ has six summands but no value more than five times, so it must be counted. Correct: $\prod_{i \geq 1} \frac{1-x^{6i}}{1-x^i}$. (The proposed function counts partitions with all parts at most 5, which by conjugation is the same as at most 5 parts: yet another question.)

Spot the error 3 (the 2!). $x^2 = 2 \cdot \frac{x^2}{2!}$, so the term adds 2 (not 1) to a_2 . Correct: $1, 1, 3, 1, 1, \dots$

Spot the error 4 (pulling $i!$ out of the sum). The summation index i cannot be moved outside the sum: “ $i! \sum_i$ ” has no meaning. The correct step keeps it inside: $\sum_i x^i = \sum_i i! \frac{x^i}{i!}$, so the coefficient of $\frac{x^i}{i!}$ is $i!$. The answer is right, the reasoning is not.

Spot the error 5 (selections, not arrangements). $(1+x)^2(1+x+x^2)^2$ is an ordinary generating function: its coefficient of x^6 is 1, the single *selection* of all six letters. HAWAII has $\frac{6!}{2!2!} = 180$ *arrangements*. Correct: $(1+x)^2(1+x+\frac{x^2}{2!})^2$.

Spot the error 6 (the initial value). $\sum_{n \geq 0} a_{n+1}x^{n+1} = f(x) - a_0$, not $f(x)$. The formula gives $a_0 = 0 \neq 1$. Correct: $(1-x)f(x) = 1 + \frac{x}{1-3x}$ and $a_n = \frac{3^n+1}{2}$.

Spot the error 7 (two different sets). A subset has exactly one smallest element, so “choose a k -subset and mark its smallest element” gives $\binom{n}{k}$ choices, not $k\binom{n}{k}$. The left side counts subsets with *any* marked member (a chair), which has k choices. With “any member” on both sides, the proof is correct.