

Tutorial 5: Assignment 2, Questions 1 to 7

Inclusion and exclusion, “exactly” and “at least”, derangements, first generating functions and palindromes of integers

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Count integer solutions with **upper bounds** by turning each bound into a “bad” condition and using the **principle of inclusion and exclusion** (PIE).
2. **Prepare** an equation before counting: shift lower bounds to 0 and **rescale** when every value is a multiple of the same number.
3. Use PIE when you are **only given the sizes** $S_0, S_1, S_2, \dots$ (and not the sets themselves).
4. Count the objects that satisfy **exactly** m or **at least** m of the conditions (E_m and L_m).
5. Count arrangements with **identical letters next to each other** by gluing them into a block.
6. Count permutations with a given number of **fixed points** using **derangements** d_n .
7. Write the **generating function** of a distribution problem: one factor per recipient, one power of x per allowed amount.
8. Count **compositions** and **palindromes of an integer** by peeling off the first and last summands.

Sections 1.1 to 1.7 review the tools with a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 solves Questions 1 to 7 of Assignment 2 in full, with the alternative methods that appeared in student solutions. Section 3 collects the classic mistakes, “spot the error” puzzles, a checklist and extra exercises. Tutorials 1 to 4 (rules of sum and product, $\binom{n}{r}$, permutations with repetition, stars and bars, shifting lower bounds, palindromes of *words*) are assumed.

Contents

1 The toolbox	2
1.1 The principle of inclusion and exclusion	2
1.2 Preparing the equation: shift, then rescale	4
1.3 “Exactly m ” and “at least m ”	6
1.4 Identical letters next to each other: glue them	7
1.5 Derangements and fixed points	8
1.6 Generating functions for distributions	10
1.7 Compositions and palindromes of an integer	11
2 The assignment problems, worked in full	13
2.1 How to write the solutions	13
2.2 Problem 1: solutions with upper bounds	14
2.3 Problem 2: Professor Bailey’s examination	16
2.4 Problem 3: did Sharon ever have lunch alone?	17
2.5 Problem 4: consecutive identical letters in ARRANGEMENT	18

2.6	Problem 5: Zelma's name cards	20
2.7	Problem 6: generating functions for pennies	21
2.8	Problem 7: palindromes of 11 and of 12	22
3	Mistakes, checklist and exercises	23
3.1	Common mistakes (and how to avoid them)	23
3.2	Checklist before you hand in	24
3.3	Exercises	24
A	Answers to the "spot the error" puzzles	25

1 The toolbox

The first half of Assignment 2 is about one big idea: **count everything, then remove what is bad, then fix what you removed twice**. Questions 1 to 5 use the principle of inclusion and exclusion in five different disguises. Questions 6 and 7 start the next chapter: generating functions. As always, every tool is explained the same way: the idea in plain words, a small example, the precise statement, a bigger example, and a recap. The examples in this section are *not* the assignment questions; those are solved in Section 2.

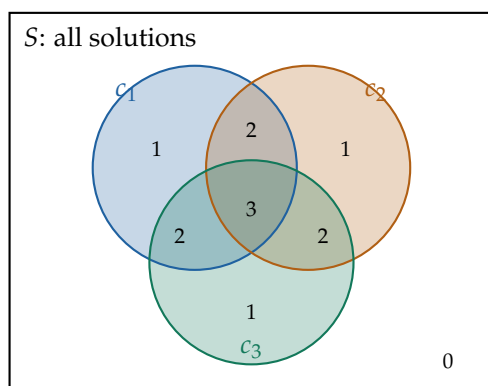
1.1 The principle of inclusion and exclusion

The intuition

Stars and bars (Tutorial 2) counts solutions with *lower* bounds easily, but *upper* bounds such as $x_i \leq 7$ are hard to build in directly. The trick is to flip the question. Call a solution **bad for** x_i if it breaks the bound, that is, $x_i \geq 8$. Bad solutions *are* easy to count: give x_i its 8 units first, then use stars and bars. So:

$$\# \text{good} = \# \text{all} - \# \text{bad}.$$

The catch: a solution can be bad for two variables at once, and then subtracting "bad for x_1 " and "bad for x_2 " removes it **twice**. Add those back once. But a solution bad for three variables was removed three times and added back three times, so remove it once more, and so on. The signs alternate: $-$, $+$, $-$, $+$, $\dots$



numbers: how many of the conditions the region satisfies

$S_0 - S_1 + S_2 - S_3$ counts a point in a region marked r exactly $\binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \binom{r}{3}$ times, which is 1 if $r = 0$ and 0 if $r \geq 1$.

Figure 1. Three conditions c_1, c_2, c_3 ("bad" properties). We want the region marked 0: the solutions that satisfy none of them. Inclusion and exclusion is a bookkeeping rule that counts that region exactly once and every other region zero times.

Simple example: $x_1 + x_2 + x_3 = 6$ with $0 \leq x_i \leq 3$

All non-negative solutions: $\binom{3+6-1}{6} = \binom{8}{2} = 28$.

Bad for x_1 ($x_1 \geq 4$): put $y_1 = x_1 - 4 \geq 0$, so $y_1 + x_2 + x_3 = 2$: $\binom{4}{2} = 6$ solutions. The same for x_2 and x_3 , so $S_1 = 3 \cdot 6 = 18$.

Bad for two variables ($x_1 \geq 4$ and $x_2 \geq 4$): the sum would be at least $8 > 6$, impossible. So $S_2 = 0$.

Good: $28 - 18 + 0 = 10$.

Check by listing. The permutations of $(3, 3, 0)$: 3; of $(3, 2, 1)$: 6; and $(2, 2, 2)$: 1. Total 10. ✓

Principle of inclusion and exclusion

Let S be a finite set with $|S| = N$, and let $c_1, \dots, c_t$ be conditions that an element of S may or may not satisfy. Write $N(c_i c_j)$ for the number of elements that satisfy c_i **and** c_j (and maybe others), and $N(\overline{c_1} \overline{c_2} \cdots \overline{c_t})$ for the number that satisfy **none** of them. Put

$$S_0 = N, \quad S_1 = \sum_i N(c_i), \quad S_2 = \sum_{i < j} N(c_i c_j), \quad \dots, \quad S_t = N(c_1 c_2 \cdots c_t).$$

Then

$$N(\overline{c_1} \overline{c_2} \cdots \overline{c_t}) = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t.$$

Symmetry shortcut. If every choice of k conditions gives the same number, then $S_k = \binom{t}{k} \cdot N(c_1 c_2 \cdots c_k)$.

Comprehensive example: different upper bounds

Count integer solutions of $x_1 + x_2 + x_3 = 12$ with $0 \leq x_1 \leq 4, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 6$.

Solution. Conditions: $c_1 : x_1 \geq 5, c_2 : x_2 \geq 6, c_3 : x_3 \geq 7$. In each case remove the forced units and count non-negative solutions of what is left (with 3 variables, $\binom{r+2}{2}$ solutions for a right-hand side r):

Term & conditions	units removed	new right-hand side	count
S_0	0	12	$\binom{14}{2} = 91$
$N(c_1), N(c_2), N(c_3)$	5, 6, 7	7, 6, 5	36, 28, 21
$N(c_1c_2), N(c_1c_3), N(c_2c_3)$	11, 12, 13	1, 0, -1	3, 1, 0
$N(c_1c_2c_3)$	18	-6	0

So $N(\overline{c_1} \overline{c_2} \cdots \overline{c_3}) = 91 - (36 + 28 + 21) + (3 + 1 + 0) - 0 = 91 - 85 + 4 = 10$.

Two slips with upper bounds

- **The bad condition is one more than the bound.** “ $x_i < 8$ ” is broken by $x_i \geq 8$, but “ $x_i \leq 8$ ” is broken by $x_i \geq 9$. Write the bad condition down explicitly for every variable.
- **A negative right-hand side means 0, not a binomial.** If the forced units exceed the total, there are no solutions. Do not plug a negative number into $\binom{r+k-1}{r}$.

PIE in one breath

- Bad condition c_i : “ x_i is too big”. Count with stars and bars after removing the forced units.
- Good = $S_0 - S_1 + S_2 - \cdots$; stop as soon as the S_k become 0, and *say why* they are 0.

1.2 Preparing the equation: shift, then rescale**The intuition**

Before using PIE, make the problem look like “non-negative integers, possibly with upper bounds”. Two moves do this.

- **Shift.** A lower bound $x_i \geq a$ is removed by giving x_i its a units first: $y_i = x_i - a \geq 0$. The right-hand side drops by a , and an upper bound $x_i \leq b$ becomes $y_i \leq b - a$.
- **Rescale.** If every value must be a multiple of 5, count in *blocks of 5* instead of single units: $y_i = x_i/5$. Everything (the total and the bounds) is divided by 5.

After the two moves, the question is again “non-negative integers with upper bounds”, and Section 1.1 takes over.

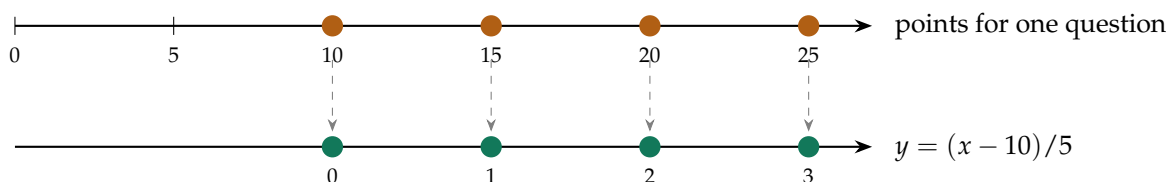


Figure 2. Shift then rescale: the four allowed values 10, 15, 20, 25 become 0, 1, 2, 3. A sum of 12 such values equal to 200 becomes a sum of 12 values in $\{0, 1, 2, 3\}$ equal to $(200 - 12 \cdot 10)/5 = 16$.

Simple example: coins in multiples of 5

In how many ways can 3 children share 40 cents if each gets a multiple of 5 cents and at least 5 cents?

Shift: give each child 5 cents, leaving 25 cents, each child now getting a non-negative multiple of 5. Rescale: $y_1 + y_2 + y_3 = 5$ with $y_i \geq 0$. Stars and bars: $\binom{7}{2} = 21$.

Shift and rescale

Let a, b, d be integers with $d \geq 1$ and $a \leq b$.

- (i) **Shift.** The map $x_i \mapsto y_i = x_i - a$ is a bijection between integers x_i with $a \leq x_i \leq b$ and integers y_i with $0 \leq y_i \leq b - a$.
- (ii) **Rescale.** If a and b are multiples of d , then $y \mapsto y/d$ is a bijection between the multiples of d in $[0, b - a]$ and the integers in $[0, (b - a)/d]$.

Applied to every variable at once, these maps turn solutions of the old equation into solutions of the new one, one to one. Shift first, then rescale: after the shift, every variable is a multiple of d that starts at 0.

Comprehensive example: shift, rescale, exclude

How many ways are there to split 60 points among 4 questions if each question is worth between 10 and 20 points and every value is a multiple of 5?

Solution. Shift by 10: $x'_1 + \dots + x'_4 = 20$ with $0 \leq x'_i \leq 10$. Rescale by 5: $y_1 + \dots + y_4 = 4$ with $0 \leq y_i \leq 2$. Bad condition $c_i : y_i \geq 3$. Then $S_0 = \binom{7}{3} = 35$, $S_1 = 4 \cdot \binom{4}{3} = 16$ (after removing 3, the right-hand side is 1), and $S_2 = 0$ since $3 + 3 > 4$. Answer: $35 - 16 = 19$.

Check by listing values of (y_i) . The permutations of $(2, 2, 0, 0)$: 6; of $(2, 1, 1, 0)$: 12; and $(1, 1, 1, 1)$: 1. Total 19. ✓

Rescaling the bounds too

The upper bound must be shifted and rescaled exactly like the variable: $x \leq 25$ becomes $x - 10 \leq 15$, then $y \leq 3$. A common slip is to rescale the total but keep the old bound ($y \leq 15$), which makes the PIE terms wrong.

Preparing in one breath

- Shift away the lower bounds (subtract from the total and from the upper bounds).
- If all values are multiples of d , divide the total and the bounds by d .

1.3 “Exactly m ” and “at least m ”

The intuition

PIE answers “how many satisfy *none* of the conditions”. Often we want more: “how many satisfy **exactly two**” or “**at least two**”. The same numbers S_k answer these too; only the coefficients change.

Why is a correction needed at all? S_2 counts each element once *for every pair* of conditions it satisfies. An element satisfying exactly 3 conditions contains $\binom{3}{2} = 3$ pairs, so S_2 counts it 3 times. To get “exactly 2”, subtract those extra counts using S_3 , then fix the over-correction using S_4 , and so on.

An element satisfying exactly r conditions	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$
is counted in S_2 this many times: $\binom{r}{2}$	0	0	1	3	6
is counted in S_3 this many times: $\binom{r}{3}$	0	0	0	1	4
is counted in S_4 this many times: $\binom{r}{4}$	0	0	0	0	1
in $S_2 - 3S_3 + 6S_4 = E_2$	0	0	1	0	0
in $S_2 - 2S_3 + 3S_4 = L_2$	0	0	1	1	1

Table 1. Why the coefficients of E_2 and L_2 (with $t = 4$ conditions) are what they are. Every column is checked by arithmetic: for $r = 4$, $6 - 3 \cdot 4 + 6 \cdot 1 = 0$ and $6 - 2 \cdot 4 + 3 \cdot 1 = 1$.

Simple example: digits

Among the 3-digit strings over $\{0, 1\}$, let c_i be “digit i is 1” ($i = 1, 2, 3$). Then $S_1 = 3 \cdot 4 = 12$, $S_2 = 3 \cdot 2 = 6$, $S_3 = 1$. Exactly two 1s: $E_2 = S_2 - \binom{3}{1}S_3 = 6 - 3 = 3$ (the strings 110, 101, 011). At least two: $L_2 = S_2 - \binom{2}{1}S_3 = 6 - 2 = 4$ (add 111). ✓

Generalized inclusion and exclusion

With $S_0, \dots, S_t$ as in Section 1.1, let E_m be the number of elements of S that satisfy **exactly** m of the conditions and L_m the number that satisfy **at least** m of them ($0 \leq m \leq t$). Then

$$E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} - \dots + (-1)^{t-m} \binom{t}{t-m} S_t,$$

$$L_m = S_m - \binom{m}{m-1}S_{m+1} + \binom{m+1}{m-1}S_{m+2} - \dots + (-1)^{t-m} \binom{t-1}{m-1} S_t \quad (m \geq 1).$$

In words: the coefficient of S_{m+j} is $(-1)^j \binom{m+j}{j}$ in E_m and $(-1)^j \binom{m+j-1}{j}$ in L_m . For $m = 0$, E_0 is the formula of Section 1.1, and $L_0 = N$. Two checks: $L_m = E_m + E_{m+1} + \dots + E_t$, and $E_0 + E_1 + \dots + E_t = N$.

Comprehensive example: fixed points of a permutation of 4

How many permutations of 1, 2, 3, 4 fix exactly one number? At least one?

Solution. Let c_i be “ i is in position i ”. Fixing k chosen numbers leaves $(4 - k)!$ arrangements, so $S_k = \binom{4}{k}(4 - k)!$: $S_1 = 24$, $S_2 = 12$, $S_3 = 4$, $S_4 = 1$. Then

$$E_1 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3 - \binom{4}{3}S_4 = 24 - 24 + 12 - 4 = 8,$$

$$L_1 = S_1 - S_2 + S_3 - S_4 = 24 - 12 + 4 - 1 = 15.$$

Check. $L_1 = 4! - (\text{permutations with no fixed point}) = 24 - 9 = 15$. ✓

Two different coefficient rows

The coefficients of E_m are $\binom{m}{0}, \binom{m+1}{1}, \binom{m+2}{2}, \dots$ (the *bottom* number grows), while those of L_m are $\binom{m-1}{m-1}, \binom{m}{m-1}, \binom{m+1}{m-1}, \dots$ (the *bottom* number is fixed at $m - 1$). Mixing them up is the most common error in Questions 4 and 5. Always check $L_m - E_m = L_{m+1}$.

Exactly and at least in one breath

- E_m : coefficients $\binom{m+j}{j}$; L_m : coefficients $\binom{m+j-1}{m-1}$; signs alternate starting with +.
- Sanity checks: $L_m = E_m + E_{m+1} + \dots + E_t$, and $\sum_m E_m = N$.

1.4 Identical letters next to each other: glue them

The intuition

“The two A’s are next to each other” is a condition about an arrangement. To count arrangements that satisfy it, **glue** the two A’s into one block $\boxed{AA}$ and arrange the blocks. The block is one object, so the word gets **one letter shorter** for every pair that is glued. Inside the block nothing can move: the two A’s are identical, so $\boxed{AA}$ has only one internal order.

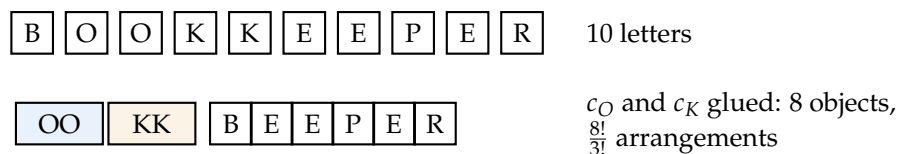


Figure 3. Gluing two pairs in BOOKKEEPER. Each glued pair removes one object. The E’s are not glued here, so they are still three identical letters among the 8 objects.

Simple example: BOOK

How many arrangements of BOOK have the two O’s together? Glue: $\boxed{OO}$, B, K are 3 distinct objects: $3! = 6$. Total arrangements $4!/2! = 12$, so $12 - 6 = 6$ have the O’s apart. Check by listing. The O’s apart means they occupy positions $\{1, 3\}$, $\{1, 4\}$ or $\{2, 4\}$, and B, K fill the other two positions in 2 orders: $3 \cdot 2 = 6$. ✓

Gluing

Consider a word with n letters in which letter L_i appears n_i times, and let c_i mean “the two copies of a letter that appears exactly twice are adjacent”. If k such doubled letters are glued, the arrangement has $n - k$ objects, the k blocks are distinct objects appearing once, and every other letter keeps its multiplicity. So

$$N(c_{i_1} \cdots c_{i_k}) = \frac{(n - k)!}{\prod (\text{multiplicities of the letters that are not glued})!}.$$

Gluing is a bijection: an arrangement in which the two A's are adjacent corresponds to exactly one arrangement of the objects with $\boxed{AA}$ as one object.

Comprehensive example: no two identical letters adjacent in AABBC

Conditions c_A, c_B . $N = \frac{5!}{2!2!} = 30$. Glue A's: $\boxed{AA}$, B, B, C gives $\frac{4!}{2!} = 12$; the same for B's: 12. Glue both: $\boxed{AA}, \boxed{BB}$, C gives $3! = 6$. So $N(\overline{c_A} \overline{c_B}) = 30 - 24 + 6 = 12$, and exactly one adjacent pair: $E_1 = S_1 - 2S_2 = 24 - 12 = 12$, and both: $E_2 = 6$. Check: $12 + 12 + 6 = 30$. ✓

Do not multiply by 2! for the inside of the block

Two *identical* A's in a block can only be “AA”. Multiplying by 2! for “the order inside the block” is wrong (it would be right for two *different* people sitting together). If you multiply by 2! and also divide by 2! for the A's, the two mistakes cancel, but the reasoning is still wrong. See Spot the error 4.

Gluing in one breath

- One glued pair = one object fewer; the block has one internal order.
- Divide by the factorials of the letters that are still repeated.

1.5 Derangements and fixed points

The intuition

Mix up n name cards and put them back at random. Card i is a **fixed point** if it lands at its own place. A **derangement** is an arrangement with *no* fixed points: nobody sits where they were supposed to. PIE with the conditions “card i is in place i ” counts derangements. And to get *exactly* m fixed points there is an even shorter route: **choose** the m cards that are in place, then **derange** the rest.

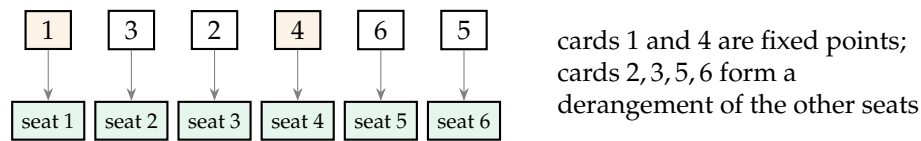


Figure 4. An arrangement of 6 cards with exactly 2 fixed points: choose the fixed cards ($\binom{6}{2}$ ways), then the other 4 cards must avoid their own seats ($d_4 = 9$ ways).

Simple example: d_3

The permutations of 1, 2, 3 with no fixed point are 231 and 312, so $d_3 = 2$. By the formula below:
 $3! \left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 6 \cdot \frac{1}{3} = 2$. ✓

Derangements

The number d_n of permutations of $1, \dots, n$ with no fixed point is ($d_0 = 1$)

$$d_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right) = \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)!.$$

n	0	1	2	3	4	5	6	7	8	9
d_n	1	0	1	2	9	44	265	1854	14833	133496

Exactly m fixed points. The number of permutations of $1, \dots, n$ with exactly m fixed points is $\binom{n}{m} d_{n-m}$. (Choose which m are fixed; the other $n - m$ must form a derangement.)

Comprehensive example: two ways to get E_1 for $n = 5$

Choose and derange. $E_1 = \binom{5}{1} d_4 = 5 \cdot 9 = 45$.

Generalized PIE. $S_k = \binom{5}{k} (5-k)! = 5!/k!$, so $S_1 = 120$, $S_2 = 60$, $S_3 = 20$, $S_4 = 5$, $S_5 = 1$, and

$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4 + 5S_5 = 120 - 120 + 60 - 20 + 5 = 45. \checkmark$$

Exactly m fixed points is not $\binom{n}{m} (n-m)!$

$\binom{n}{m} (n-m)!$ fixes m chosen cards but lets the others go anywhere, *including* their own seats. That is $N(c_{i_1} \dots c_{i_m})$ summed, i.e. S_m , which counts each arrangement with $r \geq m$ fixed points $\binom{r}{m}$ times.

Derangements in one breath

- $d_n = n! \sum_{k=0}^n (-1)^k / k!$; memorise $d_4 = 9$, $d_5 = 44$, $d_6 = 265$.
- Exactly m fixed points: $\binom{n}{m} d_{n-m}$. At least m : add these for $m, m+1, \dots, n$, or use L_m .

1.6 Generating functions for distributions

The intuition

A generating function is a **bookkeeping device**. For each person (or variable), write a polynomial whose *exponents* are the amounts that person is allowed to receive: if a child may get 0, 1, 2, ... cents, write $1 + x + x^2 + \dots$. Multiply the polynomials of all the people. Expanding the product means choosing one term from each factor, which is choosing one amount for each person, and the exponents **add up**. So the coefficient of x^n is exactly the number of ways to hand out n in total.

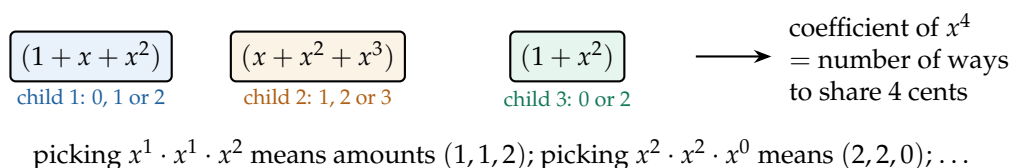


Figure 5. One factor per child, one power of x per allowed amount. Every way of choosing one term from each factor with exponents adding up to 4 is one distribution of 4 cents.

Simple example: the product of Figure 5

Expanding, $(1 + x + x^2)(x + x^2 + x^3)(1 + x^2) = x + 2x^2 + 4x^3 + 4x^4 + 4x^5 + 2x^6 + x^7$. So there are 4 ways to share 4 cents: (2, 2, 0), (1, 3, 0), (1, 1, 2), (0, 2, 2). ✓

Generating functions

The (ordinary) **generating function** of a sequence $a_0, a_1, a_2, \dots$ is $f(x) = \sum_{n \geq 0} a_n x^n$, and $[x^n]f(x)$ denotes the coefficient a_n .

(i) **Distributions.** The number of integer solutions of $x_1 + \dots + x_k = n$ with $x_i \in A_i$ is the coefficient of x^n in $\prod_{i=1}^k (\sum_{a \in A_i} x^a)$.

(ii) **Useful identities.**

$$1 + x + \dots + x^r = \frac{1 - x^{r+1}}{1 - x}, \quad \frac{1}{1 - x} = \sum_{n \geq 0} x^n, \quad \frac{1}{(1 - x)^k} = \sum_{n \geq 0} \binom{k + n - 1}{n} x^n.$$

(iii) **Factoring out the minimum.** $x^a + x^{a+1} + \dots = x^a(1 + x + \dots) = \frac{x^a}{1 - x}$.

When the question asks for “the generating function”, the answer is the function, not the number. Writing the coefficient you would need is a good habit.

Comprehensive example: 20 candies, 4 children

Find the generating function for distributing 20 identical candies among 4 children so that each child gets at least 2 and the first child gets at most 6, and find the count.

Solution. Child 1: $x^2 + x^3 + \dots + x^6$. Children 2, 3, 4: $x^2 + x^3 + \dots$ (a finite version up to x^{14} is also correct, since nobody can get more than $20 - 3 \cdot 2 = 14$). So

$$f(x) = (x^2 + \dots + x^6) (x^2 + x^3 + \dots)^3 = x^8 \frac{1 - x^5}{1 - x} \cdot \frac{1}{(1 - x)^3} = \frac{x^8(1 - x^5)}{(1 - x)^4}.$$

The count is $[x^{20}]f = [x^{12}] \frac{1 - x^5}{(1 - x)^4} = \binom{15}{12} - \binom{10}{7} = 455 - 120 = 335$. Notice that the factor $1 - x^5$ does the inclusion and exclusion for us.

Two slips with generating functions

- **Shifting twice.** If you give the oldest child 10 cents *and* write the factor $x^{10} + x^{11} + \dots$, you have counted the 10 cents twice. Either shift (and use target 25 with factor $1 + x + \dots$) or keep $x^{10} + \dots$ with target 35. Never both.
- **Answering with the number.** “Determine the generating function” asks for $f(x)$.

Generating functions in one breath

- One factor per recipient; its exponents are the allowed amounts; the answer is $[x^n]$ of the product.
- $\frac{1}{(1-x)^k}$ has coefficients $\binom{k+n-1}{n}$; a minimum a pulls out x^a .

1.7 Compositions and palindromes of an integer**The intuition**

A **composition** of n is a way to write n as an *ordered* sum of positive integers: $4 = 1 + 3$ and $4 = 3 + 1$ are different compositions. A **palindrome** of n is a composition that reads the same forwards and backwards, such as $1 + 2 + 1$ or $2 + 2$.

This is not the same as a palindrome *word* from Tutorial 4. Here the “letters” are the summands, and they must add up to n . The key move: in a palindrome the **first and last summands are equal**, so strip them off; what is left in the middle is again a palindrome, of a smaller number.

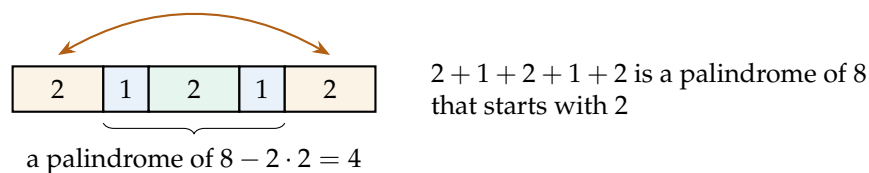


Figure 6. A palindrome of 8 starting with 2 must also end with 2; the middle $1 + 2 + 1$ is a palindrome of 4. Conversely, $2 + (\text{any palindrome of } 4) + 2$ is a palindrome of 8 starting with 2.

Simple example: palindromes of 4 and 5

Palindromes of 4: 4, $1 + 2 + 1$, $2 + 2$, $1 + 1 + 1 + 1$. That is $4 = 2^2$.

Palindromes of 5: 5, $1 + 3 + 1$, $2 + 1 + 2$, $1 + 1 + 1 + 1 + 1$. That is $5 = 2^2$. Both are $2^{\lfloor n/2 \rfloor}$.

Compositions and palindromes

- (i) The number of compositions of $n \geq 1$ is 2^{n-1} . (Write n as n ones; each of the $n - 1$ gaps either gets a “+” or not.)
- (ii) The number $p(n)$ of palindromes of $n \geq 1$ is $p(n) = 2^{\lfloor n/2 \rfloor}$. (A palindrome is decided by its “left half”, which is a composition of any $k \leq n/2$, and the middle summand, if any, is then forced.) It is convenient to set $p(0) = 1$ (the empty composition).
- (iii) **Peeling.** For $1 \leq s$ and $2s < n$, the palindromes of n that start with s are exactly $s + Q + s$ where Q is a palindrome of $n - 2s$. So there are $p(n - 2s) = 2^{\lfloor (n-2s)/2 \rfloor}$ of them. If $2s = n$ there is exactly one $(s + s)$, and if $2s > n$ the only candidate is the one-summand composition n , which starts with s only when $s = n$.

Comprehensive example: palindromes of 9 by first summand

$p(9) = 2^4 = 16$. Starting with 1: $p(7) = 8$. With 2: $p(5) = 4$. With 3: $p(3) = 2$. With 4: $p(1) = 1$. With 9 (the one-summand palindrome): 1. With 5, 6, 7, 8: none, since $2s > 9$ and $s \neq 9$. Total $8 + 4 + 2 + 1 + 1 = 16$. ✓

Composition, not word

Over the 26 letters there are $26^{\lceil n/2 \rceil}$ palindrome *words* of length n (Tutorial 4). A palindrome *of the integer* n is a composition, and there are $2^{\lfloor n/2 \rfloor}$ of them. Read which one the question means: “palindromes of 11” means compositions of the number 11.

Palindromes of integers in one breath

- $p(n) = 2^{\lfloor n/2 \rfloor}$.
- Starts with s (and $2s < n$): peel off s at both ends, $p(n - 2s)$ remain.

The toolbox at a glance

Situation	Count / method	Signal words
Upper bounds on variables	PIE with $c_i : x_i \geq b_i + 1$	" $x_i < 8$ ", " ≤ 25 "
Lower bounds, multiples of d	shift, then divide by d	"at least 10", "multiple of 5"
Only the S_k are given	$S_0 - S_1 + S_2 - \dots$	"every pair 16 times"
Exactly / at least m conditions	E_m, L_m	"exactly two", "at least two"
Identical letters adjacent	glue into a block	"consecutive identical letters"
Fixed points of a permutation	$\binom{n}{m} d_{n-m}$	"where they were supposed to be"
Distribution, "generating function"	one factor per recipient	"determine the generating function"
Palindromes of an integer	$2^{\lfloor n/2 \rfloor}$, peel the ends	"palindromes of 11"

2 The assignment problems, worked in full

2.1 How to write the solutions

All seven questions are counting questions, so the counting template from Tutorial 1 is the frame. For the PIE questions there is a second template: the PIE table. The assignment asks you to **show all the steps**: every S_k must be justified, and every S_k you set to 0 needs a one-line reason.

Writing template for a counting solution

Statement. [Restate the question precisely: what objects, what makes one "valid", what counts as "the same".]

Solution. Set-up. [Say what you are counting and prepare it: shift, rescale, glue.]

Count. [Go step by step. After each factor name the rule. If a step has cases, list them, explain why the list is complete, and add.]

Answer. [Give the expression and its numerical value.]

Check (optional but recommended). [A small case you can list, or a second method giving the same number.]

Writing template for inclusion and exclusion

1. Universe. ["Let S be the set of ...; then $N = |S| = \dots$ "]

2. Conditions. ["For $1 \leq i \leq t$, let c_i be the condition ...". Say what you want in terms of the conditions: $N(\overline{c_1} \overline{c_2} \dots \overline{c_t})$, or E_m , or L_m .]

3. The numbers S_k . [Compute $N(c_{i_1} \dots c_{i_k})$ and explain how. If they are all equal, write $S_k = \binom{t}{k} N(c_1 \dots c_k)$. Say from which k on the S_k are 0, and why.]

4. Combine. [Write the formula (PIE, E_m or L_m) with the numbers plugged in, then evaluate.]

5. Answer and check.

Phrasebook: new sentences for this tutorial

Where	What to write
Conditions	"For $1 \leq i \leq 4$, let c_i denote the condition $x_i \geq 8$. We want $N(\overline{c_1} \overline{c_2} \cdots \overline{c_4})$."
One condition	"If $x_1 \geq 8$, put $y_1 = x_1 - 8 \geq 0$; then $y_1 + x_2 + x_3 + x_4 = 11$, which has $\binom{14}{3}$ non-negative solutions."
Vanishing terms	"Three variables that are each at least 8 sum to at least $24 > 19$, so $S_3 = S_4 = 0$."
Symmetry	"By symmetry each of the $\binom{4}{2}$ pairs gives the same count, so $S_2 = \binom{4}{2} \binom{6}{3}$."
Shift and rescale	"Since every value is a multiple of 5, put $y_i = (x_i - 10)/5$; then $0 \leq y_i \leq 3$ and $\sum y_i = 16$."
Gluing	"Treat the two A's as a single block AA; this leaves 10 objects, among which R, N, E each appear twice."
Exactly m	"By the generalized principle of inclusion and exclusion, $E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$."
Fixed points	"Choose the 4 women who are seated correctly in $\binom{10}{4}$ ways; the other 6 must form a derangement, in d_6 ways."
Generating function	"Each child receives 0, 1, 2, ..., 35 cents, recorded by the factor $1 + x + \cdots + x^{35}$; the answer is the coefficient of x^{35} in ..."
Palindromes	"A palindrome of 11 that starts with 1 also ends with 1, and removing both leaves a palindrome of 9."

2.2 Problem 1: solutions with upper bounds

Problem 1 (Exercises 8.1, #6)

Determine how many integer solutions there are to $x_1 + x_2 + x_3 + x_4 = 19$, if (a) $0 \leq x_i$ for all $1 \leq i \leq 4$; (b) $0 \leq x_i < 8$ for all $1 \leq i \leq 4$; (c) $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 6$, $3 \leq x_3 \leq 7$, $3 \leq x_4 \leq 8$.

How to think about it. Part (a) is stars and bars and gives S_0 for part (b). Part (b) is Section 1.1 with four identical bounds (symmetry). Part (c) first shifts x_3, x_4 (Section 1.2), then uses PIE with four *different* bounds, so the S_k must be computed term by term.

Model solution to Problem 1(a)

Solution (what you write)

Statement. Count the non-negative integer solutions of $x_1 + x_2 + x_3 + x_4 = 19$.

Solution. By stars and bars (19 stars, 3 bars), the number is $\binom{4+19-1}{19} = \binom{22}{19} = \binom{22}{3}$.

Answer. $\boxed{\binom{22}{3} = 1540}$.

Commentary (what it does)

Non-negative, not positive: the formula is $\binom{n+k-1}{n}$.

$$\binom{22}{3} = \frac{22 \cdot 21 \cdot 20}{6}.$$

Model solution to Problem 1(b)

Solution (what you write)

1. Universe. Let S be the set of non-negative integer solutions of $x_1 + x_2 + x_3 + x_4 = 19$. By (a), $N = S_0 = \binom{22}{3} = 1540$.

2. Conditions. For $1 \leq i \leq 4$ let c_i be the condition $x_i \geq 8$. A solution has all $x_i < 8$ exactly when it satisfies none of the c_i , so we want $N(\overline{c_1} \overline{c_2} \cdots \overline{c_4})$.

3. The numbers S_k . *One condition.* If $x_1 \geq 8$, put $y_1 = x_1 - 8 \geq 0$; then $y_1 + x_2 + x_3 + x_4 = 11$, with $\binom{14}{3} = 364$ solutions. By symmetry $N(c_i) = 364$ for each i , so $S_1 = \binom{4}{1} \binom{14}{3} = 1456$. *Two conditions.* If $x_i, x_j \geq 8$, remove 16 units: the remaining sum is 3, with $\binom{6}{3} = 20$ solutions, so $S_2 = \binom{4}{2} \binom{6}{3} = 120$. *Three or more.* Three variables that are each at least 8 add up to at least $24 > 19$, so $S_3 = S_4 = 0$.

4. Combine. By the principle of inclusion and exclusion,

$$N(\overline{c_1} \overline{c_2} \cdots \overline{c_4}) = S_0 - S_1 + S_2 = 1540 - 1456 + 120 = 204.$$

Answer. $\boxed{\binom{22}{3} - \binom{4}{1} \binom{14}{3} + \binom{4}{2} \binom{6}{3} = 204}.$

Commentary (what it does)

"< 8" is broken by "≥ 8".

Symmetry: the four bounds are the same, so every k -set of conditions gives the same count.

The reason for $S_3 = 0$ is part of the solution.

Model solution to Problem 1(c)

Solution (what you write)

Set-up. Put $y_3 = x_3 - 3$ and $y_4 = x_4 - 3$. The problem becomes: count integer solutions of

$$x_1 + x_2 + y_3 + y_4 = 13,$$

$$0 \leq x_1 \leq 5, \quad 0 \leq x_2 \leq 6, \quad 0 \leq y_3 \leq 4, \quad 0 \leq y_4 \leq 5.$$

The shift is a bijection, so the number of solutions does not change.

Universe and conditions. S : non-negative solutions of the new equation, $N = \binom{16}{3} = 560$. Conditions $c_1 : x_1 \geq 6$, $c_2 : x_2 \geq 7$, $c_3 : y_3 \geq 5$, $c_4 : y_4 \geq 6$. We want $N(\overline{c_1} \overline{c_2} \cdots \overline{c_4})$.

Commentary (what it does)

Shift the upper bounds too: $x_3 \leq 7$ becomes $y_3 \leq 4$.

Each bad condition is "bound + 1".

The numbers S_k . Removing the forced units and counting with $\binom{r+3}{3}$ for a remaining sum r :

term	r	count	term	r	count
$N(c_1)$	7	$\binom{10}{3} = 120$	$N(c_1c_2)$	0	1
$N(c_2)$	6	$\binom{9}{3} = 84$	$N(c_1c_3)$	2	$\binom{5}{3} = 10$
$N(c_3)$	8	$\binom{11}{3} = 165$	$N(c_1c_4)$	1	$\binom{4}{3} = 4$
$N(c_4)$	7	$\binom{10}{3} = 120$	$N(c_2c_3)$	1	4
			$N(c_2c_4)$	0	1
			$N(c_3c_4)$	2	10
S_1		489	S_2		30

No symmetry here: the bounds are different, so all 4 + 6 terms are computed.

The smallest three forced amounts are 5, 6, 6.

Any three conditions force at least $5 + 6 + 6 = 17 > 13$ units, so $S_3 = S_4 = 0$.

Combine. $N(\overline{c_1} \overline{c_2} \cdots \overline{c_4}) = 560 - 489 + 30 = 101$.

Answer. 101.

A second method: generating functions

Part (c) is also the coefficient of x^{19} in $(1 + \cdots + x^5)(1 + \cdots + x^6)(x^3 + \cdots + x^7)(x^3 + \cdots + x^8)$, that is, of x^{13} in $\frac{(1 - x^6)(1 - x^7)(1 - x^5)(1 - x^6)}{(1 - x)^4}$. Expanding the numerator and reading off coefficients reproduces exactly the PIE table above: each product of $-x^a$ terms is one term $N(c_{i_1} \cdots c_{i_k})$. This is the bridge to Problem 6.

2.3 Problem 2: Professor Bailey's examination

Problem 2 (Exercises 8.1, #10)

Professor Bailey has just completed writing the final examination for his course in advanced engineering mathematics. This examination has 12 questions, whose total value is to be 200 points. In how many ways can Professor Bailey assign the 200 points if each question must count for at least 10, but not more than 25, points and the point value for each question is to be a multiple of 5?

How to think about it. Section 1.2 then Section 1.1: shift by 10, rescale by 5 (Figure 2), then PIE with 12 identical bounds.

Model solution to Problem 2

Solution (what you write)

Statement. Count the 12-tuples $(x_1, \dots, x_{12})$ with $x_1 + \dots + x_{12} = 200$, each $x_i \in \{10, 15, 20, 25\}$.

Set-up. Put $y_i = (x_i - 10)/5$. Then $y_i \in \{0, 1, 2, 3\}$ and $\sum y_i = (200 - 120)/5 = 16$. This is a bijection, so we count integer solutions of $y_1 + \dots + y_{12} = 16$ with $0 \leq y_i \leq 3$.

PIE. S : non-negative solutions, $N = \binom{12+16-1}{16} = \binom{27}{11}$. Condition $c_i : y_i \geq 4$. If k given variables are each ≥ 4 , removing $4k$ units leaves $\binom{12+(16-4k)-1}{11} = \binom{27-4k}{11}$ solutions, so $S_k = \binom{12}{k} \binom{27-4k}{11}$ for $k \leq 4$. For $k \geq 5$, $4k \geq 20 > 16$, so $S_k = 0$.

Combine.

$$\begin{aligned} \text{Answer} &= \binom{27}{11} - \binom{12}{1} \binom{23}{11} + \binom{12}{2} \binom{19}{11} \\ &\quad - \binom{12}{3} \binom{15}{11} + \binom{12}{4} \binom{11}{11} \\ &= 13\,037\,895 - 16\,224\,936 + 4\,988\,412 - 300\,300 + 495. \end{aligned}$$

Answer. 1 501 566 ways.

Commentary (what it does)

The questions are distinct (question 1, question 2, ...), so order matters: tuples, not multisets.

Shift: $200 - 12 \cdot 10 = 80$. Rescale: $80/5 = 16$, and $15/5 = 3$.

Writing the general S_k once saves four separate computations.

$$\begin{aligned} \binom{23}{11} &= 1\,352\,078, \quad \binom{19}{11} = 75\,582, \\ \binom{15}{11} &= 1365. \end{aligned}$$

The intermediate terms are larger than the answer; that is normal for PIE.

Same answer, generating function form

The answer is $[x^{16}] \left(\frac{1-x^4}{1-x} \right)^{12} = [x^{16}] (1-x^4)^{12} (1-x)^{-12}$. Expanding $(1-x^4)^{12}$ by the binomial theorem gives exactly the five terms above with their signs.

2.4 Problem 3: did Sharon ever have lunch alone?

Problem 3 (Exercises 8.1, #20)

At a 12-week conference in mathematics, Sharon met seven of her friends from college. During the conference she met each friend at lunch 35 times, every pair of them 16 times, every trio eight times, every foursome four times, each set of five twice, and each set of six once, but never all seven at once. If she had lunch every day during the 84 days of the conference, did she ever have lunch alone?

How to think about it. Section 1.1 in its purest form: we are *given* the numbers $N(c_{i_1} \cdots c_{i_k})$ and only need to combine them. The universe is the set of 84 lunches.

Model solution to Problem 3**Solution (what you write)**

Universe and conditions. Let S be the set of the 84 lunches, so $N = 84$. For $1 \leq i \leq 7$ let c_i be the condition “friend i is at the lunch”. Sharon is alone at a lunch exactly when it satisfies none of the c_i , so we want $N(\overline{c_1} \overline{c_2} \cdots \overline{c_7})$.

The numbers S_k . Every k -set of friends was met the same number of times, so $S_k = \binom{7}{k} \cdot N(c_1 \cdots c_k)$:

k	0	1	2	3	4	5	6	7
$N(c_1 \cdots c_k)$	84	35	16	8	4	2	1	0
$\binom{7}{k}$	1	7	21	35	35	21	7	1
S_k	84	245	336	280	140	42	7	0

Combine. $N(\overline{c_1} \overline{c_2} \cdots \overline{c_7}) = 84 - 245 + 336 - 280 + 140 - 42 + 7 - 0 = 0$.

Answer. $N(\overline{c_1} \overline{c_2} \cdots \overline{c_7}) = 0$.

No, Sharon never had lunch alone.

Commentary (what it does)

“Met friend i 35 times” means $N(c_i) = 35$: lunches with friend i , possibly with others too.

Symmetry is given by the question: “every pair”, “every trio”, ...

Group as $(84 + 336 + 140 + 7) - (245 + 280 + 42) = 567 - 567$.

Answer the yes/no question in words.

What else the numbers tell us

With the same S_k and Section 1.3 we can find how many lunches had *exactly* one friend: $E_1 = \sum_{j \geq 0} (-1)^j \binom{1+j}{j} S_{1+j} = 245 - 672 + 840 - 560 + 210 - 42 + 0 = 21$. Since $E_0 = 0$ and $E_1 = 21$, the remaining 63 lunches had at least two friends.

2.5 Problem 4: consecutive identical letters in ARRANGEMENT**Problem 4 (Exercises 8.2, #2)**

(a) In how many ways can the letters in ARRANGEMENT be arranged so that there are exactly two pairs of consecutive identical letters? At least two pairs of consecutive identical letters? (b) Answer part (a), replacing two with three.

How to think about it. Count the letters first: A, R, N, E appear twice each and G, M, T once, 11 letters in all. There are 4 possible “pairs of consecutive identical letters”, one per doubled letter. Glue (Section 1.4) to get the S_k , then use E_m and L_m (Section 1.3).

Model solution to Problem 4, the numbers S_k **Solution (what you write)**

Universe. S : the arrangements of A, A, R, R, N, N, E, E, G, M, T. Then $N = S_0 = \frac{11!}{(2!)^4} = 2\,494\,800$.

Conditions. c_1 : the two A's are adjacent; c_2 : the two R's are adjacent; c_3 : the two N's are adjacent; c_4 : the two E's are adjacent.

Commentary (what it does)

Permutations with repetition (Tutorial 1).

There are only 4 conditions, so $t = 4$.

The numbers S_k . If k of the doubled letters are glued into blocks, there are $11 - k$ objects, and the $4 - k$ doubled letters that are not glued still appear twice each. So $N(c_{i_1} \cdots c_{i_k}) = \frac{(11 - k)!}{(2!)^{4-k}}$, the same for every k -set, and

$$\begin{aligned} S_1 &= \binom{4}{1} \frac{10!}{(2!)^3} = 1\,814\,400, & S_2 &= \binom{4}{2} \frac{9!}{(2!)^2} = 544\,320, \\ S_3 &= \binom{4}{3} \frac{8!}{2!} = 80\,640, & S_4 &= \binom{4}{4} 7! = 5040. \end{aligned}$$

A block of two identical letters has one internal order (Section 1.4).

Model solution to Problem 4(a) and (b)

Solution (what you write)

(a) Exactly two. By the generalized principle of inclusion and exclusion,

$$\begin{aligned} E_2 &= S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 \\ &= 544\,320 - 241\,920 + 30\,240 = \boxed{332\,640}. \end{aligned}$$

(a) At least two.

$$\begin{aligned} L_2 &= S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4 \\ &= 544\,320 - 161\,280 + 15\,120 = \boxed{398\,160}. \end{aligned}$$

(b) Exactly three. $E_3 = S_3 - \binom{4}{1} S_4 = 80\,640 - 20\,160 = \boxed{60\,480}$.

(b) At least three. $L_3 = S_3 - \binom{3}{2} S_4 = 80\,640 - 15\,120 = \boxed{65\,520}$.

Check. $E_4 = S_4 = 5040$, and $L_3 = E_3 + E_4 = 60\,480 + 5040 = 65\,520$ ✓, $L_2 = E_2 + L_3 = 332\,640 + 65\,520 = 398\,160$ ✓.

Commentary (what it does)

Coefficients $\binom{2+j}{j}$: 1, 3, 6.

Coefficients $\binom{2+j-1}{1}$: 1, 2, 3.

$\binom{3+1-1}{2} = \binom{3}{2} = 3$.

Always run this check: it catches a wrong coefficient immediately.

Number of adjacent identical pairs	0	1	2	3	4	total
Arrangements of ARRANGEMENT	1 149 120	947 520	332 640	60 480	5040	2 494 800

Table 2. The full distribution, computed with $E_0, \dots, E_4$ and confirmed by a computer listing all 2 494 800 arrangements.
Row sum = $11!/2^4$.

2.6 Problem 5: Zelma's name cards

Problem 5 (Exercises 8.2, #6)

Zelma is having a luncheon for herself and nine of the women in her tennis league. On the morning of the luncheon she places name cards at the ten places at her table and then leaves to run a last-minute errand. Her husband, Herbert, comes home from his morning tennis match and unfortunately leaves the back door open. A gust of wind scatters the ten name cards. In how many ways can Herbert replace the ten cards at the places at the table so that exactly four of the ten women will be seated where Zelma had wanted them? In how many ways will at least four of them be seated where they were supposed to be?

How to think about it. The 10 places are distinct and so are the 10 cards, so a replacement is a permutation of 10 objects, and “seated where Zelma wanted” is a fixed point (Section 1.5). Two methods: choose-and-derange, or the generalized PIE with 10 conditions. Both appeared in student solutions.

Model solution to Problem 5, exactly four (choose and derange)

Solution (what you write)

Statement. Count the permutations of the 10 cards over the 10 places with exactly 4 fixed points.

Solution. Choose the 4 women who are seated correctly: $\binom{10}{4} = 210$ ways. The other 6 cards must be placed on the other 6 places so that none of them is at its own place: that is a derangement of 6 objects, $d_6 = 6! \left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right) = 265$ ways. By the rule of product: $210 \cdot 265$.

Answer. $\boxed{\binom{10}{4} d_6 = 55\,650}.$

Commentary (what it does)

Zelma counts as one of the ten women.

Each arrangement with exactly 4 fixed points is produced once: its fixed points determine the chosen 4.

Model solution to Problem 5, at least four

Solution (what you write)

Method 1 (add the exact counts). The number with exactly j fixed points is $\binom{10}{j} d_{10-j}$, so

$$\begin{aligned} L_4 &= \binom{10}{4} d_6 + \binom{10}{5} d_5 + \binom{10}{6} d_4 + \binom{10}{7} d_3 + \binom{10}{8} d_2 + \binom{10}{9} d_1 + \binom{10}{10} d_0 \\ &= 55\,650 + 11\,088 + 1890 + 240 + 45 + 0 + 1 = 68\,914. \end{aligned}$$

Method 2 (generalized PIE). With c_i : “card i is at place i ”, fixing k chosen cards leaves $(10 - k)!$ arrangements, so $S_k = \binom{10}{k} (10 - k)! = \frac{10!}{k!}$. Then

$$\begin{aligned} L_4 &= \sum_{j=0}^6 (-1)^j \binom{3+j}{3} S_{4+j} \\ &= 151\,200 - 120\,960 + 50\,400 - 14\,400 \\ &\quad + 3150 - 560 + 84 = 68\,914. \end{aligned}$$

Commentary (what it does)

$d_1 = 0$: nine fixed cards force the tenth to be fixed too.

$S_4 = 151\,200$, $S_5 = 30\,240$, $S_6 = 5040$, $S_7 = 720$, $S_8 = 90$, $S_9 = 10$, $S_{10} = 1$.

Answer. 68 914.

Two methods, one number: a strong check.

Exactly four, by the generalized PIE

$$E_4 = \sum_{j=0}^6 (-1)^j \binom{4+j}{j} S_{4+j} = 151\,200 - 151\,200 + 75\,600 - 25\,200 + 6\,300 - 1\,260 + 210 = 55\,650.$$

Factor out $S_4 = \frac{10!}{4!}$: $E_4 = \frac{10!}{4!} \sum_{j=0}^6 \frac{(-1)^j}{j!} = \binom{10}{4} \cdot 6! \sum_{j=0}^6 \frac{(-1)^j}{j!} = \binom{10}{4} d_6$. So the two methods are the same computation, written in two ways.

2.7 Problem 6: generating functions for pennies

Problem 6 (Exercises 9.1, #2)

Determine the generating function for the number of ways to distribute 35 pennies (from an unlimited supply) among five children if (a) there are no restrictions; (b) each child gets at least 1¢; (c) each child gets at least 2¢; (d) the oldest child gets at least 10¢; and (e) the two youngest children must each get at least 10¢.

How to think about it. Section 1.6: one factor per child. Since the total is 35, no child can get more than 35 (or less, under restrictions), so finite factors are exact; the infinite forms are equally correct and easier to work with. Do **not** shift and change the factors at the same time.

Model solution to Problem 6

Solution (what you write)

Set-up. Let x_i be the number of pennies of child i , child 1 the oldest and children 4, 5 the two youngest. The number of distributions is the number of solutions of $x_1 + \cdots + x_5 = 35$ with the given restrictions, which is $[x^{35}]$ of the product of one factor per child.

Generating functions.

$$(a) \quad f(x) = (1 + x + \cdots + x^{35})^5 = \frac{1}{(1-x)^5}$$

$$(b) \quad f(x) = (x + x^2 + \cdots + x^{31})^5 = \frac{x^5}{(1-x)^5}$$

$$(c) \quad f(x) = (x^2 + x^3 + \cdots + x^{27})^5 = \frac{x^{10}}{(1-x)^5}$$

$$(d) \quad f(x) = (x^{10} + \cdots + x^{35})(1 + \cdots + x^{25})^4 = \frac{x^{10}}{(1-x)^5}$$

$$(e) \quad f(x) = (1 + \cdots + x^{15})^3 (x^{10} + \cdots + x^{25})^2 = \frac{x^{20}}{(1-x)^5}$$

In the second form of each line, the factors are infinite: the extra terms $x^{36}, x^{37}, \dots$ never contribute to the coefficient of x^{35} .

Commentary (what it does)

Name the children so (d) and (e) are unambiguous.

(b), (c): if the others get at least 1 (or 2) each, nobody gets more than $35 - 4 = 31$ (or $35 - 8 = 27$).

(d): only the oldest child's factor changes.

(e): children 1, 2, 3 get at most $35 - 20 = 15$; children 4, 5 between 10 and $35 - 10 = 25$.

Answer. The generating functions above; in each case the number of distributions is the coefficient of x^{35} .

Any upper limit that is at least the true maximum gives a correct factor.

Going one step further: the actual numbers

Using $\frac{1}{(1-x)^5} = \sum_n \binom{n+4}{4} x^n$: (a) $[x^{35}] \frac{1}{(1-x)^5} = \binom{39}{4} = 82\,251$; (b) $[x^{30}] = \binom{34}{4} = 46\,376$; (c) and (d) $[x^{25}] = \binom{29}{4} = 23\,751$; (e) $[x^{15}] = \binom{19}{4} = 3\,876$. These match stars and bars after shifting, as they must. Notice that (c) and (d) have the same generating function: both force 10 pennies in total before the free distribution starts.

2.8 Problem 7: palindromes of 11 and of 12

Problem 7 (Exercises 9.2, #20)

- (a) How many palindromes of 11 start with 1? with 2? with 3? with 4?
 (b) How many palindromes of 12 start with 1? with 2? with 3? with 4?

How to think about it. Section 1.7: these are palindromic *compositions*. Peel off the first and last summands; in all eight cases $2s < n$, so the middle is a non-empty palindrome of $n - 2s$.

Model solution to Problem 7

Solution (what you write)

Key fact. Let $s \geq 1$ with $2s < n$. A palindrome of n that starts with s also ends with s , and removing these two summands leaves a palindrome of $n - 2s$. Conversely, for every palindrome Q of $n - 2s$, $s + Q + s$ is a palindrome of n that starts with s . So the palindromes of n starting with s are in bijection with the palindromes of $n - 2s$, and there are $p(n - 2s) = 2^{\lfloor (n-2s)/2 \rfloor}$ of them.

(a) $n = 11$:

starts with	1	2	3	4
middle is a palindrome of	9	7	5	3
count	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$

(b) $n = 12$:

starts with	1	2	3	4
middle is a palindrome of	10	8	6	4
count	$2^5 = 32$	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$

Commentary (what it does)

$2s < n$ ensures the first and last summands are two different summands.

$p(m) = 2^{\lfloor m/2 \rfloor}$ is Example 9.14 of the textbook.

For $s = 4$: $4 + 3 + 4$ and $4 + 1 + 1 + 1 + 4$.

For $s = 4$: $4 + 4 + 4$, $4 + 2 + 2 + 4$, $4 + 1 + 2 + 1 + 4$, $4 + 1 + 1 + 1 + 1 + 4$.

Where all the palindromes of 11 go

$p(11) = 2^5 = 32$. Starting with 1, 2, 3, 4, 5 gives $16 + 8 + 4 + 2 + 1 = 31$, and the one-summand palindrome 11 is the 32nd. (Starting with 5: only $5 + 1 + 5$.) For 12: $32 + 16 + 8 + 4 + 2 + 1 = 63$ for $s = 1, \dots, 6$ (the last is $6 + 6$), plus the palindrome 12: $64 = 2^6$. ✓ These totals were also

confirmed by computer.

3 Mistakes, checklist and exercises

3.1 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **Wrong bad condition.** " $x_i < 8$ " is broken by $x_i \geq 8$; " $x_i \leq 5$ " by $x_i \geq 6$.
2. **Forgetting to shift the bounds.** In Problem 1(c), after $y_3 = x_3 - 3$ the bound is $y_3 \leq 4$, not 7.
3. $S_k = 0$ **without a reason.** Write the one line: "three such variables already exceed 19".
4. **Assuming symmetry when the bounds differ.** In Problem 1(c), $N(c_1) \neq N(c_2)$.
5. **Mixing up E_m and L_m coefficients.** Check $L_m - E_m = L_{m+1}$.
6. $\binom{10}{4} \cdot 6!$ **for "exactly four".** The other six must avoid their own seats: d_6 , not $6!$.
7. **Double shifting in generating functions.** Either change the target or change the factor, not both.
8. **Palindrome words instead of palindromes of integers.** "Palindromes of 11" are compositions of 11.

Spot the error 1: Problem 1(b)

"Solution." The bad condition is $x_i > 8$. If $x_1 \geq 9$, the rest sum to 10: $\binom{13}{3}$ solutions. Two variables ≥ 9 leave 1: $\binom{4}{3}$. So the answer is $\binom{22}{3} - 4\binom{13}{3} + 6\binom{4}{3} = 1540 - 1144 + 24 = 420$.

Where is the mistake? (Answer in Appendix A.)

Spot the error 2: Problem 2

"Solution." Each question gets at least 10, so $200 - 120 = 80$ points remain, and each $x_i \leq 15$. Rescaling by 5: $y_1 + \dots + y_{12} = 16$ with $y_i \leq 15$. No variable can exceed 15 when the total is 16 unless one of them is 16, so the answer is $\binom{27}{11} - 12$.

Where is the mistake? (Answer in Appendix A.)

Spot the error 3: Problem 3

"Solution." Sharon met her friends $7 \cdot 35 = 245$ times in total, and $245 > 84$, so she always had company.

Where is the mistake? (Answer in Appendix A.)

Spot the error 4: Problem 4

"Solution." Glue the two A's into a block. The two A's inside the block can be ordered in $2!$ ways, so $N(c_1) = \frac{2! \cdot 10!}{(2!)^4}$.

Where is the mistake? (Answer in Appendix A.) Hint: list the orders of the block.

Spot the error 5: Problem 5

“Solution.” Choose the 4 women who sit correctly in $\binom{10}{4}$ ways; the other 6 can be placed in $6!$ ways. Answer: $\binom{10}{4} \cdot 6! = 151\,200$.

Where is the mistake? (Answer in Appendix A.)

Spot the error 6: Problem 6(d)

“Solution.” Give the oldest child 10 pennies first, leaving 25. The oldest now gets at least 10 more, so the generating function is $(x^{10} + x^{11} + \dots)(1 + x + \dots)^4$ and the answer is the coefficient of x^{25} .

Where is the mistake? (Answer in Appendix A.)

Spot the error 7: Problem 7(a)

“Solution.” A palindrome of 11 is decided by its first half, so there are 26^6 of them, and those starting with 1 are 26^5 .

Where is the mistake? (Answer in Appendix A.)

3.2 Checklist before you hand in**Checklist**

- ☐ I named the **universe** S and computed N .
- ☐ Every condition c_i is written down, and it is the **bad** event (bound $+1$).
- ☐ I shifted lower bounds **and** the matching upper bounds; I rescaled when all values are multiples of d .
- ☐ Each S_k is justified; symmetry is used only when the bounds are equal.
- ☐ Every $S_k = 0$ has a one-line reason.
- ☐ For “exactly” / “at least” I used the right coefficients and checked $L_m = E_m + L_{m+1}$.
- ☐ Glued blocks of identical letters have **one** internal order.
- ☐ For fixed points, the non-fixed objects form a **derangement**.
- ☐ A generating function answer gives $f(x)$ and says which coefficient is needed.
- ☐ For palindromes of integers, I peeled off the first and last summands and used $2^{\lfloor m/2 \rfloor}$.
- ☐ Every answer is **boxed** (expression and number).

3.3 Exercises

Write each solution in full, using the templates.

- E1.** How many integer solutions does $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ have with $0 \leq x_i \leq 10$ for all i ?
- E2.** How many integer solutions does $x_1 + x_2 + x_3 = 15$ have with $2 \leq x_1 \leq 6$, $0 \leq x_2 \leq 8$, $1 \leq x_3 \leq 7$? Check your answer with a generating function.
- E3.** A 6-question quiz is worth 60 points; each question is worth between 5 and 15 points and a multiple of 5. In how many ways can the points be assigned?
- E4.** In a class of 40 students, 25 study French, 20 Spanish, 10 German; 10 study French and Spanish, 5 French and German, 4 Spanish and German, and 2 study all three. How many study none of the three? Exactly one?

- E5. How many arrangements of AABBBCCD have no two identical letters adjacent? Exactly one pair of adjacent identical letters? Check that your $E_0, \dots, E_3$ add up to the total.
- E6. How many permutations of $1, \dots, 8$ have exactly 3 fixed points? At least 3? Check that $\sum_{m=0}^8 \binom{8}{m} d_{8-m} = 8!$.
- E7. Prove that $d_n = (n-1)(d_{n-1} + d_{n-2})$ for $n \geq 2$ by looking at where the card of person 1 goes.
- E8. Find the generating function, and then the number, of ways to distribute 24 pennies among 4 children if each child gets an even number of pennies and the youngest gets at least 6.
- E9. How many palindromes of 13 start with 1? with 3? with 6? with 7? Check that the counts for $s = 1, \dots, 6$, plus the one-summand palindrome, add up to $p(13)$.
- E10. Show that the number of compositions of n is 2^{n-1} in two ways: the gap argument, and induction on n using the first summand.

A Answers to the “spot the error” puzzles

Spot the error 1 (the bound). “ $x_i < 8$ ” allows $x_i = 0, \dots, 7$; it is broken already by $x_i = 8$. The bad condition is $x_i \geq 8$, not $x_i > 8$. With $x_i > 8$ the solutions having some $x_i = 8$ (such as $(8, 8, 3, 0)$) were never removed. Correct: $1540 - 4\binom{14}{3} + 6\binom{6}{3} = 204$.

Spot the error 2 (rescaling the bound). After dividing by 5 the upper bound $x_i - 10 \leq 15$ becomes $y_i \leq 3$, not $y_i \leq 15$. The total was rescaled but the bound was not. Correct: PIE with $c_i : y_i \geq 4$, giving 1 501 566.

Spot the error 3 (a sum is not a union). 245 counts a lunch with three friends three times. A large total says nothing about whether some lunch had no friend: in principle all 245 meetings could have happened on a few days. Inclusion and exclusion is needed to get $N(\overline{c_1} \overline{c_2} \cdots \overline{c_7}) = 0$.

Spot the error 4 (identical letters in a block). The two A’s are identical, so the block AA has only one order. Multiplying by $2!$ counts every arrangement twice. Here the error is hidden because the student also divided by $(2!)^4$ (four repeated letters) instead of $(2!)^3$ (only R, N, E are still repeated after gluing), and the two errors cancel. Correct: $N(c_1) = \frac{10!}{(2!)^3}$.

Spot the error 5 (fixed points among the rest). Placing the other six cards in $6!$ ways allows some of them to land at their own places, so arrangements with 5, 6, $\dots$ fixed points are counted, and counted several times. The six must form a derangement: $\binom{10}{4} d_6 = 55\,650$. (The number 151 200 is S_4 .)

Spot the error 6 (double shift). The 10 pennies are given twice: once by changing the target to 25, and again by the factor $x^{10} + \dots$. This counts distributions where the oldest gets at least 20. Use either target 35 with $x^{10} + x^{11} + \dots$, or target 25 with $1 + x + \dots$ for the oldest.

Spot the error 7 (wrong kind of palindrome). “Palindromes of 11” are compositions of the integer 11, not words over the alphabet. There are $2^5 = 32$ of them, and $p(9) = 16$ start with 1.