

Tutorial 4: Assignment 1, Questions 8 to 14

Positive solutions, systems of equations, induction with inequalities, Fibonacci sums, ascents and palindromes

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Count **positive** integer solutions of $x_1 + \cdots + x_k = n$ with $\binom{n-1}{k-1}$, and use the **symmetry** $\binom{N}{a} = \binom{N}{N-a}$ to decide when two binomial coefficients are equal.
2. Count solutions of a **system** of equations by splitting the variables into independent groups and multiplying.
3. Prove an **upper bound** by induction when the step splits a problem into two halves (merge sort), and keep inequalities pointing the right way.
4. **Grow** a statement about two terms into a statement about n terms (the triangle inequality), using grouping and the inductive hypothesis.
5. Prove identities about the **Fibonacci numbers** by induction, using the recurrence at exactly the right step.
6. Count permutations by their number of **ascents**, and derive a recurrence by **inserting the largest element** into a smaller permutation.
7. Count **palindromes** by noticing that the first half of the word decides the whole word.

Sections 1.1 to 1.7 review the tools with a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 solves Questions 8 to 14 of Assignment 1 in full, with the alternative methods that appeared in student solutions. Section 3 collects the classic mistakes, “spot the error” puzzles, a checklist and extra exercises. Tutorials 1 to 3 (counting rules, $\binom{n}{r}$, stars and bars, slack variables, proof by induction) are assumed.

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1 The toolbox

The second half of Assignment 1 has two flavours. Questions 8, 9 and 14 are **counting** questions that reuse stars and bars and the rule of product in a new way. Questions 10, 11 and 12 are **proofs by induction**. Question 13 mixes both: you count small cases by hand, then explain a recurrence. As before, every tool is explained the same way: the idea in plain words, a small example, the precise statement, a bigger example, and a recap. The examples in this section are *not* the assignment questions; those are solved in Section 2.

1.1 Positive solutions and the symmetry of binomial coefficients

The intuition

Stars and bars (Tutorial 2) counts *non-negative* solutions: a variable may be 0. If every variable must be **at least 1**, give each variable one star first. What is left is an ordinary non-negative problem. There is also a direct picture: line up the n stars and put the $k - 1$ bars **in the gaps between stars**, at most one bar per gap, so that no part is empty.

The second idea of this section: $\binom{N}{a}$ counts the ways to *choose* a objects, which is the same as choosing the $N - a$ objects to *leave out*. So $\binom{N}{a} = \binom{N}{N-a}$, and two different lower numbers can give the same binomial coefficient.



Figure 1. The positive solution $(2, 3, 2)$ of $x_1 + x_2 + x_3 = 7$. The bars may only go in the $n - 1 = 6$ gaps between stars (the small ticks), never at an end and never two in the same gap, so every part gets at least one star.

Simple example: $x_1 + x_2 + x_3 = 5$ with all $x_i \geq 1$

Shift. Put $y_i = x_i - 1 \geq 0$. Then $y_1 + y_2 + y_3 = 2$, which has $\binom{3+2-1}{2} = \binom{4}{2} = 6$ non-negative solutions.

Gaps. 5 stars have 4 inner gaps; choose 2 of them for the bars: $\binom{4}{2} = 6$.

Check by listing. $(3, 1, 1), (1, 3, 1), (1, 1, 3), (2, 2, 1), (2, 1, 2), (1, 2, 2)$: six solutions. ✓

Positive solutions and symmetry

(i) **Positive solutions.** Let $k \geq 1$. The number of solutions of $x_1 + \cdots + x_k = n$ in *positive* integers is

$$\binom{n-1}{k-1} \quad \text{if } n \geq k, \quad \text{and } 0 \text{ if } n < k.$$

Equivalently, after the shift $y_i = x_i - 1$: $\binom{k+(n-k)-1}{n-k} = \binom{n-1}{n-k}$, the same number.

(ii) **Symmetry.** For integers $0 \leq a \leq N$: $\binom{N}{a} = \binom{N}{N-a}$.

(iii) **When are two binomials equal?** For integers $0 \leq a, b \leq N$:

$$\binom{N}{a} = \binom{N}{b} \iff a = b \text{ or } a + b = N.$$

(In row N of Pascal's triangle the entries increase up to the middle and then decrease, and each value appears at most twice, in mirror positions.)

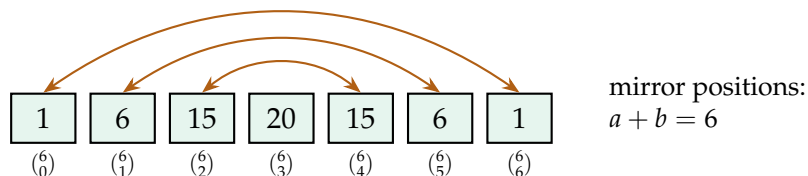


Figure 2. Row 6 of Pascal's triangle. Equal values sit in mirror positions a and $6 - a$; apart from that, all values are different.

Comprehensive example: same number of positive solutions

For which positive integer n do $x_1 + \cdots + x_5 = n$ and $y_1 + \cdots + y_{12} = n$ have the same (non-zero) number of positive integer solutions?

Solution. By (i), the counts are $\binom{n-1}{4}$ and $\binom{n-1}{11}$ (for $n \geq 12$ both are non-zero). By (iii) they are equal exactly when $4 = 11$ (impossible) or $4 + 11 = n - 1$, that is, $n = 16$.

Check. $\binom{15}{4} = 1365 = \binom{15}{11}$. ✓

Two slips with positive solutions

- Using the non-negative formula $\binom{n+k-1}{n}$ for a question that says *positive*. “Positive” means ≥ 1 ; “non-negative” means ≥ 0 .
- Concluding “ $\binom{N}{a} = \binom{N}{b}$ forces $a = b$ ”. Equal binomials also come from mirror positions $a + b = N$; forgetting this makes it look as if there is no answer.

Positive solutions in one breath

- $x_1 + \cdots + x_k = n$, all $x_i \geq 1$: $\binom{n-1}{k-1}$ (bars in the $n - 1$ inner gaps).
- $\binom{N}{a} = \binom{N}{b}$ iff $a = b$ or $a + b = N$.

1.2 Systems of equations: split into independent parts

The intuition

Two equations that share variables look harder than one. But if one equation contains *only some* of the variables of the other, subtract: $x_1 + \cdots + x_7 = 37$ and $x_1 + x_2 + x_3 = 6$ together say exactly the same as $x_1 + x_2 + x_3 = 6$ and $x_4 + \cdots + x_7 = 31$. Now the two equations have **no variable in common**, so a solution of the system is **a solution of the first part together with a solution of the second part**: rule of product.

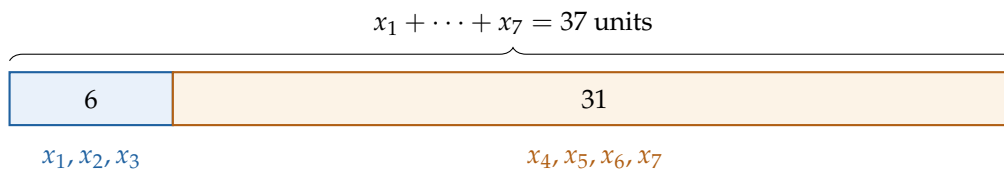


Figure 3. The second equation reserves 6 of the 37 units for x_1, x_2, x_3 . The other 31 units go to $x_4, \dots, x_7$, independently of how the first 6 were shared.

Simple example: a tiny system

Count non-negative solutions of $x_1 + x_2 + x_3 + x_4 = 5$ and $x_1 + x_2 = 2$. Subtracting gives $x_3 + x_4 = 3$. The first part $x_1 + x_2 = 2$ has 3 solutions, the second part $x_3 + x_4 = 3$ has 4. By the rule of product: $3 \cdot 4 = 12$.

Check by listing the first part. For each of $(2,0), (1,1), (0,2)$ the second part can be $(3,0), (2,1), (1,2), (0,3)$: $3 \cdot 4 = 12$. ✓

Independent parts

Suppose the variables split into disjoint groups $G_1, \dots, G_t$, and the system is equivalent to one equation per group: $\sum_{x \in G_j} x = r_j$ for $j = 1, \dots, t$, together with conditions that each involve only one group. Then

$$\text{\#solutions of the system} = \prod_{j=1}^t \text{\#solutions of the equation for } G_j.$$

The equivalence must go *both ways*: every solution of the system satisfies the new equations, and every choice of solutions for the groups satisfies the original system. Subtracting one equation from another and adding it back gives exactly this.

Comprehensive example: a three-part system

How many non-negative integer solutions have $x_1 + \dots + x_6 = 20$, $x_1 + x_2 = 5$ and $x_3 + x_4 = 8$, if moreover $x_5 \geq 2$?

Solution. Subtracting, the system is equivalent to $x_1 + x_2 = 5$, $x_3 + x_4 = 8$, $x_5 + x_6 = 7$, three equations on disjoint groups. The condition $x_5 \geq 2$ only involves the third group: put $y_5 = x_5 - 2$, so $y_5 + x_6 = 5$. The counts are $\binom{6}{1} = 6$, $\binom{9}{1} = 9$ and $\binom{6}{1} = 6$. By the rule of product: $6 \cdot 9 \cdot 6 = 324$.

Product, not sum

The two parts are chosen *together* (one solution for the first part **and** one for the second), so the counts are multiplied. Adding them would count “a solution of the first part **or** a solution of the second part”, which is not a solution of the system at all.

Systems in one breath

- Subtract to make the groups of variables disjoint; then count each group separately and multiply.
- Extra conditions (> 0 , ≥ 2) are handled inside the group they belong to.

1.3 Induction for upper bounds: split in half**The intuition**

Some algorithms solve a problem of size 2^{k+1} by solving **two problems of size 2^k** and then **combining** the answers. Merge sort is the classic example: sort each half, then merge the two sorted halves. Induction on k fits this perfectly: the inductive hypothesis tells us how much the two halves cost, and we only have to add the cost of the combining step.

Because we prove an **upper bound** (“at most”), we may use $\leq$ at every step, and at the end it is enough to land *below* the target. Landing on $(k+1)2^{k+1} - 1$ when the target is $(k+1)2^{k+1}$ is a success, not a failure.

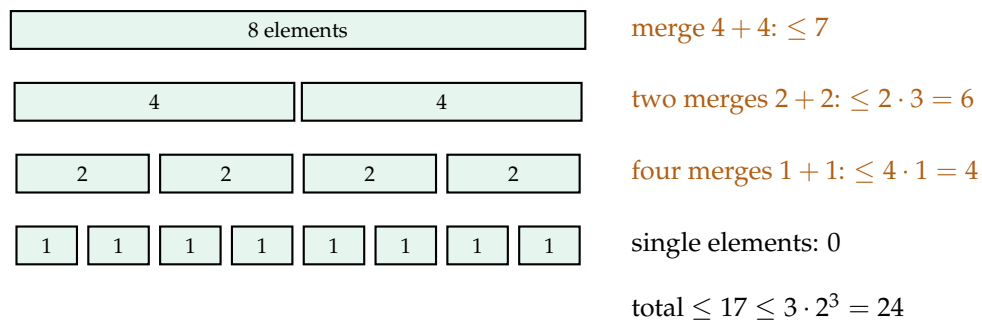


Figure 4. Merge sort on $2^3 = 8$ elements. There are $n = 3$ levels of merging, and each level costs fewer than $2^n = 8$ comparisons. This is where the bound $n \cdot 2^n$ comes from; induction makes the picture a proof.

Simple example: four numbers

Sort $\{7, 2, 9, 4\}$. Split into $\{7, 2\}$ and $\{9, 4\}$. Each pair is sorted with 1 comparison: $(2, 7)$ and $(4, 9)$. Merge: compare 2 and 4 (take 2), compare 7 and 4 (take 4), compare 7 and 9 (take 7), then 9 is left. That is $3 = 2 + 2 - 1$ comparisons. Total $1 + 1 + 3 = 5 \leq 2 \cdot 2^2 = 8$. ✓

Proving a bound $T(2^n) \leq f(n)$ by induction

Let $T(m)$ be the number of steps needed for an input of size m . To prove $T(2^n) \leq f(n)$ for all $n \geq 0$:

- (i) **Base case.** Check $T(2^0) = T(1) \leq f(0)$.
- (ii) **Inductive hypothesis.** For an arbitrary $k \geq 0$, assume that *every* input of size 2^k can be handled in at most $f(k)$ steps.
- (iii) **Inductive step.** Split an input of size 2^{k+1} into two inputs of size 2^k , apply the hypothesis to *each* half, add the cost $c(k)$ of combining, and show

$$2f(k) + c(k) \leq f(k+1).$$

For merge sort, $f(n) = n \log n$ and $c(k) = 2^k + 2^k - 1 = 2^{k+1} - 1$.

Comprehensive example: the exact worst case

Suppose merging always costs *exactly* $m + r - 1$ comparisons, so $C(1) = 0$ and $C(2^{k+1}) = 2C(2^k) + 2^{k+1} - 1$. Prove that $C(2^n) = (n-1)2^n + 1$ for all $n \geq 0$.

Solution. *Base case.* $C(1) = 0$ and $(0-1)2^0 + 1 = 0$. *Inductive step.* Assume $C(2^k) = (k-1)2^k + 1$. Then

$$C(2^{k+1}) = 2((k-1)2^k + 1) + 2^{k+1} - 1 = (k-1)2^{k+1} + 2^{k+1} + 1 = k2^{k+1} + 1,$$

which is the formula for $n = k + 1$. Since $(n-1)2^n + 1 \leq n2^n$ for $n \geq 0$, this is below the bound of Question 10.

Check. $C(2) = 1$, $C(4) = 5$, $C(8) = 17$, and $(3-1) \cdot 8 + 1 = 17$, as in Figure 4. ✓

Three slips in bound proofs

- **Base case at the wrong place.** If the claim is for all $n \geq 0$, the base case is $n = 0$ (one element, 0 comparisons), not $n = 1$.
- **A hypothesis that is too weak.** The IH must be about *every* set of size 2^k , because it is applied to two different halves.
- **Starting from the goal.** Writing the inequality you want and simplifying it to " $0 \leq 1$ " only proves something if every step can be reversed. Safer: start from the cost and push it *up* to the bound.

Bounds by induction in one breath

- Split, apply the IH to each half, add the combining cost, then show the total is $\leq$ the bound for $k + 1$.
- For merge sort: $2 \cdot k2^k + (2^{k+1} - 1) = (k+1)2^{k+1} - 1 \leq (k+1)2^{k+1}$.

1.4 From two terms to n terms

The intuition

Many facts are first proved for **two** objects: $|x + y| \leq |x| + |y|$, $|xy| = |x||y|$, “the union of two finite sets is finite”. To extend them to n objects, **group** the first k objects into one block: $x_1 + \cdots + x_k + x_{k+1} = (x_1 + \cdots + x_k) + x_{k+1}$. Now it is a two-object situation again. Use the two-term fact once, and the inductive hypothesis takes care of the block.

$$\boxed{x_1 + x_2 + \cdots + x_k} + \boxed{x_{k+1}} \quad \text{two-term fact on } a + b, \text{ then the IH on the block } a$$

one block: a
 b

Figure 5. Grouping turns $k + 1$ terms into two terms. The two-term fact is used with $a = x_1 + \cdots + x_k$ and $b = x_{k+1}$.

Simple example: three numbers

Show $|x + y + z| \leq |x| + |y| + |z|$ from the two-term inequality. Group: $|x + y + z| = |(x + y) + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$. The first $\leq$ is the two-term inequality with $a = x + y$, $b = z$; the second is the two-term inequality again, for x, y . Induction does exactly this, once for each new term.

Extending a two-term property by induction

Suppose a property holds for any two objects: $\Phi(a, b)$ for all a, b . To prove the n -term version for all $n \geq 2$:

- (i) **Base case** $n = 2$: this is the two-term fact.
- (ii) **Inductive hypothesis**: for an arbitrary $k \geq 2$, the statement holds for *any* k objects.
- (iii) **Inductive step**: given $k + 1$ objects, group the first k (associativity), apply the two-term fact to the block and the last object, then apply the IH to the block.

The order of the two justifications matters for the inequality: the two-term fact first, the IH second.

Comprehensive example: absolute value of a product

Given $|ab| = |a||b|$ for all real a, b , prove $|x_1 x_2 \cdots x_n| = |x_1| |x_2| \cdots |x_n|$ for all $n \geq 2$ and all reals $x_1, \dots, x_n$.

Solution. *Base case* $n = 2$: given. *Inductive step*. Let $k \geq 2$ and assume the statement for any k reals. For reals $x_1, \dots, x_{k+1}$:

$$|x_1 \cdots x_k x_{k+1}| = |(x_1 \cdots x_k) x_{k+1}| = |x_1 \cdots x_k| |x_{k+1}| \stackrel{\text{IH}}{=} |x_1| \cdots |x_k| |x_{k+1}|.$$

The first equality is associativity, the second the two-term fact with $a = x_1 \cdots x_k$, $b = x_{k+1}$. By induction the statement holds for all $n \geq 2$.

Where the two-term fact is used

The inequality $|a + b| \leq |a| + |b|$ is applied with $a = x_1 + \cdots + x_k$, which is *one real number*. Say so. A step that jumps from $|x_1 + \cdots + x_{k+1}|$ straight to $|x_1| + \cdots + |x_{k+1}|$ “by the IH” uses the statement for $k + 1$ terms, which is what we are trying to prove.

Two terms to n terms in one breath

- Group the first k terms, use the two-term fact once, then the IH on the group.
- Base case $n = 2$; name both justifications in the step.

1.5 Fibonacci numbers and recursive definitions**The intuition**

The Fibonacci numbers are defined by a **starting point and a rule**: $F_0 = 0$, $F_1 = 1$, and each later number is the sum of the two before it. Proofs about them by induction have one special ingredient: somewhere in the inductive step you must **use the recurrence** $F_m = F_{m-1} + F_{m-2}$, and you should say so. For sums $F_0 + \cdots + F_n$, the step always starts the same way: **split off the last term**.

n	0	1	2	3	4	5	6	7	8
F_n	0	1	1	2	3	5	8	13	21
$F_0 + \cdots + F_n$	0	1	2	4	7	12	20	33	54
$F_{n+2} - 1$	0	1	2	4	7	12	20	33	54

Table 1. The first Fibonacci numbers and their partial sums. The last two rows agree: this is Question 12.

Simple example: computing with the recurrence

$F_2 = F_1 + F_0 = 1$, $F_3 = F_2 + F_1 = 2$, $F_4 = 3$, $F_5 = 5$. Notice that the recurrence read backwards gives $F_m = F_{m+2} - F_{m+1}$, for instance $F_3 = F_5 - F_4 = 5 - 3 = 2$. This backwards form gives a second proof of Question 12 (see Section 2).

Fibonacci numbers

The Fibonacci numbers are defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Splitting off the last term. For every $k \geq 0$: $\sum_{i=0}^{k+1} F_i = \left(\sum_{i=0}^k F_i \right) + F_{k+1}$.

Using the recurrence at the top. For every $k \geq 0$: $F_{k+2} + F_{k+1} = F_{k+3}$ (the recurrence with $n = k+3 \geq 2$).

Comprehensive example: sum of the odd-indexed terms

Prove that $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$ for all $n \geq 1$.

Solution. *Base case* $n = 1$. The left side is $F_1 = 1$ and the right side is $F_2 = 1$. *Inductive step.* Let $k \geq 1$ and assume $F_1 + F_3 + \cdots + F_{2k-1} = F_{2k}$. The sum for $n = k + 1$ has one more term, F_{2k+1} :

$$F_1 + \cdots + F_{2k-1} + F_{2k+1} \stackrel{\text{IH}}{=} F_{2k} + F_{2k+1} = F_{2k+2} = F_{2(k+1)},$$

by the recurrence. By the principle of mathematical induction the identity holds for all $n \geq 1$.

Check ($n = 3$): $F_1 + F_3 + F_5 = 1 + 2 + 5 = 8 = F_6$. ✓

Two slips with Fibonacci induction

- **The wrong last term.** Going from $n = k$ to $n = k + 1$ adds the term with index $k + 1$, here F_{k+1} , not F_{k+2} . Write the new sum out with its last term before you simplify.
- **A different starting convention.** Some books start with $F_0 = 1, F_1 = 1$. State the definition you use. The course uses $F_0 = 0, F_1 = 1$.

Fibonacci induction in one breath

- Write $\sum_{i=0}^{k+1} = \sum_{i=0}^k + (\text{last term})$, use the IH, then the recurrence.
- Name the recurrence when you use it: “since $F_{k+3} = F_{k+2} + F_{k+1}$ ”.

1.6 Ascents, descents and inserting the largest element**The intuition**

Read a permutation from left to right and look at each pair of neighbours: it either goes **up** (an ascent) or **down** (a descent). A permutation of $1, \dots, m$ has $m - 1$ neighbouring pairs, so ascents and descents add up to $m - 1$.

To count permutations of $1, \dots, m$ with k ascents, build them from permutations of $1, \dots, m - 1$ by **inserting** m , the largest number. Because m is larger than everybody, what happens to the number of ascents depends only on *where* it goes, and there are only four kinds of places.

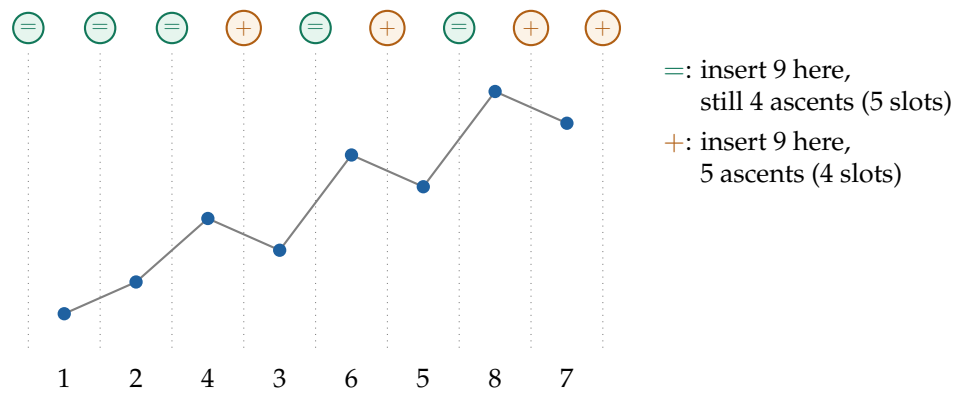


Figure 6. The permutation $p = 12436587$ drawn as a path: up steps are ascents, down steps are descents. Inserting 9 at the start or inside an **up** step keeps the number of ascents; inserting it inside a **down** step or at the end adds one ascent.

Simple example: the permutations of 1, 2, 3

0 ascents	1 ascent	2 ascents
321	132, 213, 231, 312	123

So the counts are 1, 4, 1, and $1 + 4 + 1 = 6 = 3!$. ✓ Reversing a permutation turns ascents into descents, which is why the row is symmetric.

Ascents, descents and the insertion rule

Let $p = p_1 p_2 \cdots p_m$ be a permutation of $1, \dots, m$. A position i ($1 \leq i \leq m-1$) is an **ascent** if $p_i < p_{i+1}$ and a **descent** if $p_i > p_{i+1}$. Let $\pi_{m,k}$ be the number of permutations of $1, \dots, m$ with exactly k ascents.

- (i) $\# \text{ascents} + \# \text{descents} = m - 1$.
- (ii) **Insertion rule.** Insert m into one of the m slots of a permutation q of $1, \dots, m-1$ (before the first entry, between two entries, or after the last). Then, since m is larger than every entry of q :

Slot	What changes	Ascents
at the start	a new descent $m > q_1$	unchanged
inside an ascent $a < b$	$a < b$ becomes $a < m > b$	unchanged
inside a descent $a > b$	$a > b$ becomes $a < m > b$	+1
at the end	a new ascent $q_{m-1} < m$	+1

- (iii) Removing m from a permutation of $1, \dots, m$ gives back q and the slot, so every permutation of $1, \dots, m$ arises from exactly one pair (q, slot) .

Comprehensive example: from 1,2,3 to 1,2,3,4

Count the permutations of 1,2,3,4 with exactly 1 ascent, by inserting 4.

Solution. A permutation of 1,2,3,4 with 1 ascent comes from a permutation q of 1,2,3 in one of two ways.

- q has 1 ascent and 4 goes to a slot that keeps the ascents: the start, or inside the one ascent: 2 slots. There are 4 such q , giving $4 \cdot 2 = 8$.
- q has 0 ascents ($q = 321$) and 4 goes to a slot that adds one: inside one of the 2 descents, or at the end: 3 slots, giving $1 \cdot 3 = 3$.

Total $8 + 3 = 11$, which matches Table 2 in Section 2.7.

Inserting the largest number does not always add an ascent

It is tempting to think “ m is the biggest, so it always creates an ascent”. It does create one ascent *into* m (unless it is first) but it also creates a descent *out of* m (unless it is last), and it destroys the pair it was inserted into. Use the table in (ii), not intuition.

Ascents in one breath

- Ascents + descents = $m - 1$.
- Insert m : ascents stay the same in $k + 1$ slots (start and the k ascents), and go up by one in the other slots (end and the descents).

1.7 Palindromes: the first half decides**The intuition**

A **palindrome** reads the same forwards and backwards: *level*, *noon*, *racecar*. Position 1 must equal position n , position 2 must equal position $n - 1$, and so on. So once the **first half** of the word is chosen, the second half is forced. For odd n the middle letter is its own mirror image, so it is free too.



Figure 7. Blue positions are chosen freely, gray positions are forced by the mirror, and the orange middle letter of an odd palindrome is free. The number of free positions is $\lceil n/2 \rceil$.

Simple example: binary palindromes of length 3

Choose positions 1 and 2 freely ($2 \cdot 2 = 4$ ways); position 3 must equal position 1. The four palindromes are 000, 010, 101, 111, and $2^{\lceil 3/2 \rceil} = 2^2 = 4$. ✓

Counting palindromes

A string $s_1s_2 \cdots s_n$ is a palindrome if $s_i = s_{n+1-i}$ for every i . Over an alphabet with q symbols, the map “keep the first $\lceil n/2 \rceil$ symbols” is a bijection from palindromes of length n onto all strings of length $\lceil n/2 \rceil$. Hence the number of palindromes of length n is

$$q^{\lceil n/2 \rceil} = \begin{cases} q^{n/2}, & n \text{ even,} \\ q^{(n+1)/2}, & n \text{ odd.} \end{cases}$$

For $n = 0$ this gives 1 (the empty string).

Comprehensive example: palindromes with a condition

How many palindromes of length 7 over the 26 lowercase letters contain the letter a at least once?

Solution. A palindrome of length 7 is decided by its first 4 letters, and it contains a if and only if those 4 letters do (every later letter copies an earlier one). So count strings of length 4 with at least one a , by the complement: $26^4 - 25^4 = 456\,976 - 390\,625 = 66\,351$.

Do not forget the middle letter

The answer $26^{n/2}$ is right for even n only. For $n = 1$ it gives $26^{1/2}$, which is not even an integer, while there are clearly 26 palindromes of length 1. Always test a formula on $n = 1$ and $n = 2$.

Palindromes in one breath

- The first $\lceil n/2 \rceil$ letters decide the palindrome: $q^{\lceil n/2 \rceil}$.
- A condition on the palindrome becomes a condition on its first half.

The toolbox at a glance

Situation	Count / method	Signal words
$x_1 + \cdots + x_k = n$, all $x_i \geq 1$	$\binom{n-1}{k-1}$	“positive integer solutions”
Two binomials are equal	$a = b$ or $a + b = N$	“for which $n \dots$ the same number”
Equations sharing variables	subtract, split, multiply	“the pair of equations”
Bound for a divide-and-conquer algorithm	induction on n , IH on each half	“bounded above by”, “ $ S = 2^n$ ”
Two-term fact to n terms	group, two-term fact, then IH	“for $n \geq 2$ and $x_1, \dots, x_n$ ”
Identity with F_n	split off last term, IH, recurrence	“Fibonacci”, “prove by induction”
Permutations by ascents	insert the largest element	“ascents”, “descents”, “ $\pi_{m,k}$ ”
Palindromes of length n	$q^{\lceil n/2 \rceil}$	“reads the same backwards”

2 The assignment problems, worked in full

2.1 How to write the solutions

The counting questions (8, 9, 13 A to E, 14) use the counting template from Tutorial 1. The proofs (10, 11, 12, 13 F) use the induction template from Tutorial 2. Both are repeated below. The assignment asks you to **show all the steps**: a correct number or a correct formula with no argument is not a solution.

Writing template for a counting solution

Statement. [Restate the question precisely: what objects, what makes one “valid”, what counts as “the same”.]

Solution. Set-up. [Say what you are counting and settle the questions: positive or non-negative? which variables are linked? which symbols are allowed?]

Count. [Go step by step. After each factor name the rule. If a step has cases, list them, explain why the list is complete, and add.]

Answer. [Give the expression and its numerical value.]

Check (optional but recommended). [A small case you can list, or a second method giving the same number.]

Writing template for a proof by induction

Statement. [Define $P(n)$ and the range: “For every integer $n \geq n_0, \dots$ ”. Say “We prove this by induction on n .”]

Base case. [Check $P(n_0)$ by direct computation. Show both sides as numbers.]

Inductive hypothesis. [“Let $k \geq n_0$ be an arbitrary integer and assume $P(k)$, that is, $\dots$ ” Write $P(k)$ out in full.]

Inductive step. [“We show $P(k+1)$, that is, $\dots$ ” Write the goal out in full. Then start from one side of the goal and transform it until you reach the other side (or the required inequality). Mark the step that uses the IH.]

Conclusion. [“By the principle of mathematical induction, $P(n)$ holds for every integer $n \geq n_0$.”]

Phrasebook: new sentences for this tutorial

Where	What to write
Positive solutions	“Since each $x_i \geq 1$, put $a_i = x_i - 1 \geq 0$; then $a_1 + \dots + a_{19} = n - 19$.”
Symmetry	“Since $\binom{N}{a} = \binom{N}{N-a}$, we have $\binom{n-1}{n-19} = \binom{n-1}{18}$.”
Systems	“Subtracting the second equation from the first, the system is equivalent to $\dots$; the two equations have no variable in common, so by the rule of product $\dots$ ”
Bounds	“By the inductive hypothesis, each half can be sorted with at most $k 2^k$ comparisons.”
Grouping	“Applying the two-term inequality with $x = x_1 + \dots + x_k$ and $y = x_{k+1}, \dots$ ”
Recurrence	“Since $F_{k+3} = F_{k+2} + F_{k+1}$ by the definition of the Fibonacci numbers, $\dots$ ”
Insertion	“Inserting m into a descent $a > b$ gives $a < m > b$: the descent is replaced by an ascent and a descent, so the number of ascents increases by one.”
Palindromes	“A palindrome is determined by its first $\lceil n/2 \rceil$ letters, and every choice of these letters gives a palindrome.”

2.2 Problem 8: two equations with the same number of solutions

Problem 8 (Exercises 1.4, #16)

For which positive integer n will the equations (A) $x_1 + x_2 + x_3 + \cdots + x_{19} = n$ and (B) $y_1 + y_2 + y_3 + \cdots + y_{64} = n$ have the same number of positive integer solutions?

How to think about it. Section 1.1 twice: write both counts as binomial coefficients with the same upper number $n - 1$, then use the symmetry rule to decide when they are equal.

Model solution to Problem 8

Solution (what you write)

Statement. Find the positive integers n for which (A) and (B) have the same number of solutions with all variables positive integers.

Solution. *Count for (A).* Since each $x_i \geq 1$, put $a_i = x_i - 1 \geq 0$. Then (A) becomes $a_1 + \cdots + a_{19} = n - 19$, which for $n \geq 19$ has $\binom{19+(n-19)-1}{n-19} = \binom{n-1}{n-19} = \binom{n-1}{18}$ solutions. *Count for (B).* In the same way, with $b_i = y_i - 1$, (B) has $\binom{n-1}{n-64} = \binom{n-1}{63}$ solutions for $n \geq 64$.

Compare. For $19 \leq n \leq 63$, (A) has at least one solution and (B) has none, so the counts differ. For $n \geq 64$ we need $\binom{n-1}{18} = \binom{n-1}{63}$. Two binomial coefficients $\binom{N}{a}$ and $\binom{N}{b}$ are equal only if $a = b$ or $a + b = N$. Since $18 \neq 63$, we need $18 + 63 = n - 1$, that is, $n = 82$.

Answer. $n = 82$, where both equations have $\binom{81}{18} = \binom{81}{63}$ positive solutions.

Commentary (what it does)

The shift is a bijection between positive solutions and non-negative solutions of the new equation. The last equality is the symmetry $\binom{N}{a} = \binom{N}{N-a}$.

Forgetting the mirror case $a + b = N$ makes it look as if there is no answer.

$\binom{81}{18} = 456\,703\,981\,505\,085\,600$, a number you do not need to compute.

A closer look at small n

For $1 \leq n \leq 18$ neither equation has a positive solution (you cannot write n as a sum of 19 positive integers if $n < 19$), so both counts are 0 and are trivially “the same”. The question clearly means the case where solutions exist, and $n = 82$ is the only such value. A careful solution mentions this in one sentence.

2.3 Problem 9: a pair of equations

Problem 9 (Exercises 1.4, #18)

- (A) How many non-negative integer solutions are there to the pair of equations $x_1 + x_2 + x_3 + \cdots + x_7 = 37$, $x_1 + x_2 + x_3 = 6$?
- (B) How many solutions in the previous part have $x_1, x_2, x_3 > 0$?

How to think about it. Section 1.2: subtract to get two equations on disjoint groups of variables, count each with stars and bars, multiply. Part (B) only changes the first group, so the second count is reused.

Model solution to Problem 9(A)

Solution (what you write)

Statement. Count the 7-tuples of non-negative integers with $x_1 + \cdots + x_7 = 37$ and $x_1 + x_2 + x_3 = 6$.

Solution. Subtracting the second equation from the first, the pair is equivalent to

$$x_1 + x_2 + x_3 = 6 \quad \text{and} \quad x_4 + x_5 + x_6 + x_7 = 31.$$

(Conversely, adding these two equations gives back $x_1 + \cdots + x_7 = 37$.) The two equations have no variable in common, so a solution of the pair is a solution of the first equation together with a solution of the second. By stars and bars, $x_1 + x_2 + x_3 = 6$ has $\binom{3+6-1}{6} = \binom{8}{6} = 28$ solutions and $x_4 + \cdots + x_7 = 31$ has $\binom{4+31-1}{31} = \binom{34}{31} = 5984$ solutions. By the rule of product the pair has $28 \cdot 5984$ solutions.

Answer. $\boxed{\binom{8}{6} \binom{34}{31} = 28 \cdot 5984 = 167\,552}.$

Commentary (what it does)

The “conversely” sentence shows the two systems have exactly the same solutions. Then the rule of product applies because the choices are independent.

Model solution to Problem 9(B)

Solution (what you write)

Statement. Count the solutions of (A) with $x_1, x_2, x_3 \geq 1$.

Solution. The condition only involves the first equation, so the second still has 5984 solutions. For the first, put $w_i = x_i - 1 \geq 0$ ($i = 1, 2, 3$): then $w_1 + w_2 + w_3 = 3$, with $\binom{3+3-1}{3} = \binom{5}{3} = 10$ solutions. By the rule of product: $10 \cdot 5984$.

Answer. $\boxed{\binom{5}{3} \binom{34}{31} = 10 \cdot 5984 = 59\,840}.$

Check (first part). The positive solutions of $x_1 + x_2 + x_3 = 6$: the 3 permutations of $(4, 1, 1)$, the 6 of $(3, 2, 1)$ and $(2, 2, 2)$: $3 + 6 + 1 = 10$. ✓

Commentary (what it does)

For integers, > 0 is the same as ≥ 1 .

Equivalently, by Section 1.1: $\binom{6-1}{3-1} = \binom{5}{2} = 10$.

2.4 Problem 10: merge sort uses at most $n \cdot 2^n$ comparisons

Problem 10 (Exercises 4.1, #20)

Let S_1 and S_2 be two sets where $|S_1| = m$, $|S_2| = r$, for $m, r \in \mathbb{Z}^+$, and the elements in each of S_1, S_2 are in ascending order. It can be shown that the elements in S_1 and S_2 can be merged into ascending order by making no more than $m + r - 1$ comparisons. Use this result to establish the following. For $n \geq 0$, let S be a set with $|S| = 2^n$. Prove that the number of comparisons needed to place the elements of S in ascending order is bounded above by $n \cdot 2^n$.

How to think about it. Section 1.3. The statement is about *every* set of size 2^n , so the induction is on n , and the step splits a set of size 2^{k+1} into two halves of size 2^k . “The number of comparisons needed” is at most the number used by *one* particular method, so describing a method (sort both halves, then merge) and bounding its cost is enough.

Model solution to Problem 10

Solution (what you write)

Statement. For $n \geq 0$, let $P(n)$ be: *every* set S with $|S| = 2^n$ can be placed in ascending order using at most $n \cdot 2^n$ comparisons. We prove $P(n)$ for all $n \geq 0$ by induction on n .

Base case ($n = 0$). A set with $2^0 = 1$ element is already in ascending order, so 0 comparisons are needed, and $0 \leq 0 \cdot 2^0 = 0$. So $P(0)$ holds.

Inductive hypothesis. Let $k \geq 0$ be an arbitrary integer and assume $P(k)$: every set of 2^k elements can be placed in ascending order with at most $k \cdot 2^k$ comparisons.

Inductive step. We show $P(k+1)$. Let $|S| = 2^{k+1}$ and split S into two disjoint sets S_1, S_2 with $|S_1| = |S_2| = 2^k$. By the inductive hypothesis, S_1 and S_2 can each be sorted with at most $k \cdot 2^k$ comparisons. By the given merging result with $m = r = 2^k$, the two sorted sets can then be merged with at most $2^k + 2^k - 1 = 2^{k+1} - 1$ comparisons. In total, S can be sorted with at most

$$\begin{aligned} k2^k + k2^k + 2^{k+1} - 1 &= k2^{k+1} + 2^{k+1} - 1 \\ &= (k+1)2^{k+1} - 1 \leq (k+1)2^{k+1} \end{aligned}$$

comparisons. This proves $P(k+1)$.

Conclusion. By the principle of mathematical induction, $P(n)$ holds for every integer $n \geq 0$.

Commentary (what it does)

“Every set” matters: the IH will be used on two different halves.

The claim starts at $n = 0$, so the base case is $n = 0$.

Sort, sort, merge: the three costs are added (rule of sum for the steps of the method). Ending at $(k+1)2^{k+1} - 1$ is fine: we only need “at most”.

How tight is the bound?

The comprehensive example of Section 1.3 shows that if every merge used the maximum number of comparisons, merge sort would use exactly $(n-1)2^n + 1$ comparisons. So the bound $n \cdot 2^n$ is off by less than 2^n : it has the right order of growth, $2^n \log_2 2^n$, which is the famous $N \log_2 N$ for $N = 2^n$ elements.

2.5 Problem 11: the triangle inequality for n numbers

Problem 11 (Exercises 4.2, #10)

For all $x \in \mathbb{R}$, $|x| = \sqrt{x^2}$, which is x if $x \geq 0$ and $-x$ if $x < 0$, and $-|x| \leq x \leq |x|$. Consequently $|x + y|^2 = (x + y)^2 = x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2 = (|x| + |y|)^2$, and so $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$. Prove that if $n \in \mathbb{Z}^+$, $n \geq 2$, and $x_1, x_2, \dots, x_n \in \mathbb{R}$, then

$$|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|.$$

How to think about it. Section 1.4: the two-term inequality is given (and proved in the question), so it is the base case. In the step, group the first k numbers into one real number.

Model solution to Problem 11

Solution (what you write)

Statement. For $n \geq 2$, let $P(n)$ be: for all $x_1, \dots, x_n \in \mathbb{R}$, $|x_1 + \dots + x_n| \leq |x_1| + \dots + |x_n|$. We prove $P(n)$ for all $n \geq 2$ by induction on n .

Base case ($n = 2$). $P(2)$ is the inequality $|x + y| \leq |x| + |y|$ with $x = x_1, y = x_2$, which is proved in the statement of the question.

Inductive hypothesis. Let $k \geq 2$ be an arbitrary integer and assume $P(k)$: for all reals $x_1, \dots, x_k$, $|x_1 + \dots + x_k| \leq |x_1| + \dots + |x_k|$.

Inductive step. We show $P(k + 1)$. Let $x_1, \dots, x_{k+1} \in \mathbb{R}$. Then

$$\begin{aligned} |x_1 + \dots + x_k + x_{k+1}| &= |(x_1 + \dots + x_k) + x_{k+1}| \\ &\leq |x_1 + \dots + x_k| + |x_{k+1}| \\ &\leq |x_1| + \dots + |x_k| + |x_{k+1}|. \end{aligned}$$

The equality is associativity of addition; the first inequality is the two-term inequality with $x = x_1 + \dots + x_k$ and $y = x_{k+1}$; the second inequality is the inductive hypothesis. This proves $P(k + 1)$.

Conclusion. By the principle of mathematical induction, $P(n)$ holds for every integer $n \geq 2$.

Commentary (what it does)

The “for all $x_1, \dots, x_n$ ” is part of $P(n)$.

It is fine to cite the given proof; you may also copy the three-line argument.

Line 1: group. Line 2: two-term fact. Line 3: IH (adding $|x_{k+1}|$ to both sides of the IH).

2.6 Problem 12: the sum of the first Fibonacci numbers

Problem 12 (Exercises 4.2, #12)

For $n \geq 0$ let F_n denote the n th Fibonacci number. Prove by induction that

$$F_0 + F_1 + F_2 + \cdots + F_n = \sum_{i=0}^n F_i = F_{n+2} - 1.$$

How to think about it. Section 1.5: split off the last term, use the IH, then the recurrence. Table 1 already shows the identity for $n \leq 8$.

Model solution to Problem 12

Solution (what you write)

Statement. With $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, let $P(n)$ be: $\sum_{i=0}^n F_i = F_{n+2} - 1$. We prove $P(n)$ for all $n \geq 0$ by induction on n .

Base case ($n = 0$). The left side is $F_0 = 0$. The right side is $F_2 - 1 = (F_1 + F_0) - 1 = 1 - 1 = 0$. So $P(0)$ holds.

Inductive hypothesis. Let $k \geq 0$ be an arbitrary integer and assume $P(k)$: $\sum_{i=0}^k F_i = F_{k+2} - 1$.

Inductive step. We show $P(k+1)$, that is, $\sum_{i=0}^{k+1} F_i = F_{k+3} - 1$. We have

$$\begin{aligned} \sum_{i=0}^{k+1} F_i &= \left(\sum_{i=0}^k F_i \right) + F_{k+1} \\ &= (F_{k+2} - 1) + F_{k+1} \\ &= (F_{k+2} + F_{k+1}) - 1 = F_{k+3} - 1, \end{aligned}$$

by the inductive hypothesis (second line) and since $F_{k+3} = F_{k+2} + F_{k+1}$ by the definition (third line). This proves $P(k+1)$.

Conclusion. By the principle of mathematical induction, $P(n)$ holds for every integer $n \geq 0$.

Commentary (what it does)

State the definition you use.

Both sides as numbers.

Write the goal with $k+3$ (not “ $(k+1) + 2$ ”) so you know where you are heading. The recurrence is used with $n = k+3 \geq 3$, so it applies.

A second proof: telescoping

The recurrence read backwards says $F_i = F_{i+2} - F_{i+1}$ for every $i \geq 0$. Adding these for $i = 0, \dots, n$, every term $F_2, \dots, F_{n+1}$ appears once with $+$ and once with $-$:

$$\sum_{i=0}^n F_i = \sum_{i=0}^n (F_{i+2} - F_{i+1}) = F_{n+2} - F_1 = F_{n+2} - 1.$$

This also explains a student solution that used $F_0 = F_1 = 1$: the sum always equals $F_{n+2} - F_1$, and $F_1 = 1$ in both conventions. The question says “prove by induction”, so the induction proof is the one to hand in.

2.7 Problem 13: ascents and descents

Problem 13 (Exercises 4.2, #18)

Consider the permutations of 1, 2, 3, 4. The permutation 1432 has one ascent, 14, and two descents, 43 and 32. The permutation 1423 has two ascents, at 14 and 23, and the one descent 42.

- A) How many permutations of 1, 2, 3 have k ascents, for $k = 0, 1, 2$?
 B) How many permutations of 1, 2, 3, 4 have k ascents, for $k = 0, 1, 2, 3$?
 C) If a permutation of 1, 2, ..., 7 has four ascents, how many descents does it have?
 D) Suppose a permutation of 1, 2, 3, ..., m has k ascents, for $0 \leq k \leq m - 1$. How many descents does it have?
 E) Consider $p = 12436587$, which has four ascents. In how many of the nine locations (at the start, end, or between two numbers) in p can we place 9 so that the result has (i) four ascents; (ii) five ascents?
 F) Let $\pi_{m,k}$ denote the number of permutations of 1, 2, ..., m with k ascents. Note how $\pi_{4,2} = 11 = 2(4) + 3(1) = (4 - 2)\pi_{3,1} + (2 + 1)\pi_{3,2}$. How is $\pi_{m,k}$ related to $\pi_{m-1,k-1}$ and $\pi_{m-1,k}$?

How to think about it. Parts A and B: list carefully (Table 2). Parts C and D: ascents and descents share the $m - 1$ neighbouring pairs. Part E: the insertion table of Section 1.6. Part F: the same table, now for a general permutation, gives the recurrence.

Ascents	Permutations of 1, 2, 3, 4	Count
0	4321	1
1	1432, 2143, 2431, 3142, 3214, 3241, 3421, 4132, 4213, 4231, 4312	11
2	1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, 4123	11
3	1234	1
		total 24 = 4!

Table 2. Problem 13(B). Reversing a permutation with 1 ascent gives one with 2 ascents (and back), which is why the two middle rows have the same size.

Model solution to Problem 13(A) to (D)

Solution (what you write)

(A) Listing the $3! = 6$ permutations: 0 ascents: 321; 1 ascent: 132, 213, 231, 312; 2 ascents: 123. So $\pi_{3,0} = 1$, $\pi_{3,1} = 4$, $\pi_{3,2} = 1$, and $1 + 4 + 1 = 6$. ✓

(B) Listing the $4! = 24$ permutations (Table 2): $\pi_{4,0} = 1$, $\pi_{4,1} = 11$, $\pi_{4,2} = 11$, $\pi_{4,3} = 1$, and $1 + 11 + 11 + 1 = 24$. ✓

(C) A permutation of 7 numbers has 6 pairs of neighbours, and each pair is either an ascent or a descent (never both, never neither, since the numbers are distinct). With 4 ascents there are $6 - 4 = 2$ descents.

Commentary (what it does)

Always add up the row and compare with $m!$.

Reversal pairs 1-ascent and 2-ascent permutations, so the middle counts are equal: $(24 - 2)/2 = 11$.

Say why each pair is exactly one of the two.

(D) In the same way, a permutation of $1, \dots, m$ has $m - 1$ pairs of neighbours, so with k ascents it has $m - 1 - k$ descents.

Model solution to Problem 13(E)

Solution (what you write)

Set-up. $p = 12436587$ has the ascents 12, 24, 36, 58 and the descents 43, 65, 87. Since 9 is larger than every entry, inserting it (Section 1.6, Figure 6):

- at the start adds the descent $9 > 1$: ascents unchanged;
- inside an ascent $a < b$ gives $a < 9 < b$: one ascent is replaced by an ascent and a descent, ascents unchanged;
- inside a descent $a > b$ gives $a < 9 > b$: one descent is replaced by an ascent and a descent, ascents +1;
- at the end adds the ascent $7 < 9$: ascents +1.

(i) Four ascents: the start and the 4 ascents: 5 locations, giving 912436587, 192436587, 129436587, 124396587, 124365987.

(ii) Five ascents: the 3 descents and the end: 4 locations, giving 124936587, 124369587, 124365897, 124365879.

Commentary (what it does)

The four kinds of slot cover all 9 locations: 1 start, 4 ascents, 3 descents, 1 end.

Check: $5 + 4 = 9$ locations. ✓

Model solution to Problem 13(F)

Solution (what you write)

Claim. For $m \geq 2$ and $0 \leq k \leq m - 1$ (with $\pi_{m-1,-1} = \pi_{m-1,m-1} = 0$),

$$\pi_{m,k} = (m - k) \pi_{m-1,k-1} + (k + 1) \pi_{m-1,k}.$$

Proof. Every permutation of $1, \dots, m$ is obtained in exactly one way by inserting m into one of the m slots of a permutation q of $1, \dots, m - 1$: removing m recovers q and the slot. By the insertion rule, the result has the same number of ascents as q if m goes at the start or inside an ascent of q , and one more ascent if m goes at the end or inside a descent of q . So a permutation with k ascents arises in exactly one of two ways.

Case 1: q has k ascents. Then m must go at the start or inside one of the k ascents: $k + 1$ slots. This gives $(k + 1) \pi_{m-1,k}$ permutations.

Case 2: q has $k - 1$ ascents. By (D), q has $(m - 2) - (k - 1) = m - k - 1$ descents. Then m must go at the end or inside one of these descents: $m - k$ slots. This gives $(m - k) \pi_{m-1,k-1}$ permutations.

The two cases are disjoint, so by the rule of sum $\pi_{m,k} = (m - k) \pi_{m-1,k-1} + (k + 1) \pi_{m-1,k}$. Check: $\pi_{4,2} = 2 \cdot \pi_{3,1} + 3 \cdot \pi_{3,2} = 2 \cdot 4 + 3 \cdot 1 = 11$. ✓

Commentary (what it does)

The convention $\pi = 0$ outside the range takes care of $k = 0$ and $k = m - 1$.

This bijection is what makes the count exact: no permutation is produced twice or missed.

q is a permutation of $m - 1$ numbers, so it has $m - 2$ pairs of neighbours.

Part (E) is this argument for one particular q : $k + 1 = 5$ and $m - k = 9 - 5 = 4$.

$m \setminus k$	0	1	2	3	4	5	sum
1	1						1
2	1	1					2
3	1	4	1				6
4	1	11	11	1			24
5	1	26	66	26	1		120
6	1	57	302	302	57	1	720

Table 3. The numbers $\pi_{m,k}$ (the Eulerian numbers), computed with the recurrence of 13(F). Each row adds up to $m!$, and each row is symmetric. For example $\pi_{5,2} = 3 \cdot 11 + 3 \cdot 11 = 66$.

What about the sum of a row?

One student solution went further and proved by induction that the recurrence gives $\sum_k \pi_{m,k} = m!$. That is a good consistency check, but it is not what (F) asks: (F) asks for the relation, and the insertion argument above both finds it and proves it. The row sums in Table 3 are a quick numerical check of the same fact.

2.8 Problem 14: palindromes

Problem 14

Find the number of palindromes of length n over the English alphabet.

How to think about it. Section 1.7 with $q = 26$. State your alphabet (the 26 letters, one case) and treat even and odd n together with $\lceil n/2 \rceil$.

Model solution to Problem 14

Solution (what you write)

Statement. Count the strings $s_1 s_2 \cdots s_n$ over the 26 letters $a, \dots, z$ with $s_i = s_{n+1-i}$ for $1 \leq i \leq n$.

Solution. The condition pairs position i with position $n + 1 - i$. For $i \leq \lceil n/2 \rceil$ these pairs cover every position: if n is even, the first $n/2$ positions are paired with the last $n/2$; if n is odd, the middle position $\frac{n+1}{2}$ is paired with itself. So a palindrome is determined by its first $\lceil n/2 \rceil$ letters, and conversely every choice of these letters, completed by the mirror rule, gives a palindrome. Each of the $\lceil n/2 \rceil$ free positions has 26 choices, so by the rule of product there are $26^{\lceil n/2 \rceil}$ palindromes.

Answer. $26^{\lceil n/2 \rceil}$, that is, $26^{n/2}$ for even n and $26^{(n+1)/2}$ for odd n .

Check. $n = 1$: 26 palindromes (every letter). $n = 2$: 26 ($aa, \dots, zz$). $n = 3$: $26^2 = 676$ (aba -type: first and middle letter free). ✓

Commentary (what it does)

Write the alphabet you assume. Uppercase letters would give 52 instead of 26.

The “conversely” sentence makes “first half $\leftrightarrow$ palindrome” a bijection.

Note that lengths $2j - 1$ and $2j$ give the same count.

A second method: a recurrence

One student solution built palindromes of length n from palindromes of length $n - 2$ by adding the same letter at both ends: $P_n = 26 P_{n-2}$ for $n \geq 2$, with $P_0 = 1$ (the empty string) and $P_1 = 26$. Unrolling gives $P_n = 26^{\lceil n/2 \rceil}$. To make this a complete proof, show $P_n = 26^{\lceil n/2 \rceil}$ by induction (two base cases, $n = 0$ and $n = 1$, because the step goes from $n - 2$ to n).

3 Mistakes, checklist and exercises

3.1 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **Non-negative instead of positive.** In Problem 8 the variables are positive; $\binom{n+18}{n}$ is the count for non-negative variables, a different question.
2. **Forgetting the mirror solution.** $\binom{n-1}{18} = \binom{n-1}{63}$ does not need $18 = 63$; it needs $18 + 63 = n - 1$.
3. **Adding instead of multiplying.** In Problem 9 a solution of the pair is a solution of the first part *and* of the second part: $28 \cdot 5984$, not $28 + 5984$.
4. **Base case at $n = 1$ for a claim about $n \geq 0$.** In Problem 10 the case $|S| = 2^0 = 1$ must be checked.
5. **Using the claim for $k + 1$ terms inside its own proof.** In Problem 11 the IH is about k terms only; group first, then use the two-term inequality, then the IH.
6. **Adding the wrong last term.** In Problem 12 the sum up to $k + 1$ adds F_{k+1} , not F_{k+2} .
7. **"The largest number always adds an ascent".** In Problem 13(E) it does so in only 4 of the 9 slots.
8. **Forgetting the middle letter.** In Problem 14, $26^{n/2}$ is not an integer for odd n .

Spot the error 1: Problem 8

"Solution." The counts are $\binom{n-1}{n-19}$ and $\binom{n-1}{n-64}$. For these to be equal we need $n - 19 = n - 64$, i.e. $19 = 64$, which is impossible. So no such n exists.

Where is the mistake? (Answer in Appendix A.)

Spot the error 2: Problem 9(A)

"Solution." The first equation $x_1 + x_2 + x_3 = 6$ has $\binom{8}{6} = 28$ solutions and the remaining equation $x_4 + \dots + x_7 = 31$ has $\binom{34}{31} = 5984$ solutions. By the rule of sum the pair has $28 + 5984 = 6012$ solutions.

Where is the mistake? (Answer in Appendix A.)

Spot the error 3: Problem 10

"Proof." Base case $n = 1$: a set of 2 elements is sorted with 1 comparison, and $1 \leq 1 \cdot 2^1 = 2$. Inductive step: exactly as in the model solution. Hence the bound holds for all $n \geq 0$.

Where is the mistake? (Answer in Appendix A.)

Spot the error 4: Problem 12

"Inductive step." $\sum_{i=0}^{k+1} F_i = \sum_{i=0}^k F_i + F_{k+2} \stackrel{\text{IH}}{=} F_{k+2} - 1 + F_{k+2} = 2F_{k+2} - 1 = F_{k+3} - 1.$

Where is the mistake? (Answer in Appendix A.) Hint: try $k = 2$.

Spot the error 5: Problem 13(E)

"Solution." Since 9 is larger than every other number, it always creates a new ascent wherever we put it. So all 9 locations give five ascents, and no location gives four ascents.

Where is the mistake? (Answer in Appendix A.) Hint: insert 9 between 1 and 2 and count.

Spot the error 6: Problem 14

"Solution." The first half of the word decides the second half, so there are $n/2$ free letters and $26^{n/2}$ palindromes.

Where is the mistake? (Answer in Appendix A.)

3.2 Checklist before you hand in**Checklist**

- ☐ I checked whether the variables are **positive** or **non-negative**, and shifted accordingly.
- ☐ When two binomial coefficients must be equal, I considered both $a = b$ and $a + b = N$.
- ☐ For a system, I **subtracted** to get disjoint groups, said why the new system is equivalent, and **multiplied**.
- ☐ In every induction proof, the **base case** matches the start of the claim ($n = 0$ or $n = 2$).
- ☐ My inductive hypothesis is written out in full, and is about **every** set / **any** k numbers.
- ☐ In the step, I wrote the goal $P(k + 1)$ before computing, and **marked** where the IH is used.
- ☐ For an upper bound, my chain of $\leq$ ends **at or below** the target.
- ☐ For Fibonacci, I split off the **correct last term** and named the recurrence.
- ☐ For ascents, I checked my counts add up to $m!$.
- ☐ For palindromes, I tested $n = 1$ and $n = 2$.
- ☐ Every answer is **boxed** (expression and number).

3.3 Exercises

Write each solution in full, using the templates.

- E1.** For which positive integer n do $x_1 + \cdots + x_{10} = n$ and $y_1 + \cdots + y_{30} = n$ have the same non-zero number of positive integer solutions?
- E2.** How many non-negative integer solutions have $x_1 + \cdots + x_6 = 20$, $x_1 + x_2 = 5$ and $x_3 + x_4 = 8$?
- E3.** How many solutions of Problem 9(A) have $x_i \geq 2$ for $4 \leq i \leq 7$?
- E4.** Prove by induction that if $C(1) = 0$ and $C(2^{k+1}) = 2C(2^k) + 2^{k+1} - 1$ for $k \geq 0$, then $C(2^n) = (n - 1)2^n + 1$ for all $n \geq 0$. (This is the comprehensive example of Section 1.3; write it with the template.)
- E5.** Using $|ab| = |a||b|$, prove that $|x_1 x_2 \cdots x_n| = |x_1| |x_2| \cdots |x_n|$ for all $n \geq 2$ and reals $x_1, \dots, x_n$.
- E6.** Prove by induction, for $n \geq 0$: (a) $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$; (b) $F_1 + F_3 + \cdots + F_{2n+1} = F_{2n+2}$.

- E7. (a) Use the recurrence of Problem 13(F) to compute $\pi_{5,k}$ for $k = 0, \dots, 4$, and check the sum. (b) Show that $\pi_{m,0} = \pi_{m,m-1} = 1$ for all $m \geq 1$. (c) Prove by induction that $\pi_{m,1} = 2^m - m - 1$ for all $m \geq 1$.
- E8. Show that $\pi_{m,k} = \pi_{m,m-1-k}$ by giving a bijection between the permutations with k ascents and those with $m - 1 - k$ ascents.
- E9. (a) How many binary strings of length 10 are *not* palindromes? (b) How many palindromes of length 7 over $a, \dots, z$ contain at least one a ? (c) How many 5-digit positive integers read the same backwards (the first digit cannot be 0)?
- E10. Prove by induction that the number P_n of palindromes of length n over 26 letters satisfies $P_n = 26^{\lceil n/2 \rceil}$, using $P_n = 26 P_{n-2}$ for $n \geq 2$. Why do you need two base cases?

A Answers to the “spot the error” puzzles

Spot the error 1 (forgetting symmetry). Equal lower numbers are one way for two binomial coefficients with the same upper number to be equal, but not the only one: $\binom{N}{a} = \binom{N}{N-a}$. Here $\binom{n-1}{n-19} = \binom{n-1}{18}$ and $\binom{n-1}{n-64} = \binom{n-1}{63}$, and these are equal when $18 + 63 = n - 1$, i.e. $n = 82$.

Spot the error 2 (sum instead of product). A solution of the pair needs *both* a solution of $x_1 + x_2 + x_3 = 6$ *and* a solution of $x_4 + \dots + x_7 = 31$, chosen independently. That is the rule of product: $28 \cdot 5984 = 167\,552$. The rule of sum would count objects that are one *or* the other.

Spot the error 3 (base case). The claim is for all $n \geq 0$, but the proof starts at $n = 1$, so $n = 0$ is never covered: the inductive step only reaches $n = 1, 2, 3, \dots$. Check $n = 0$ (one element, $0 \leq 0$ comparisons) instead, or as well.

Spot the error 4 (wrong last term). The sum up to $k + 1$ adds the term F_{k+1} , not F_{k+2} . The last equality $2F_{k+2} - 1 = F_{k+3} - 1$ is also false: for $k = 2$ it says $2 \cdot 3 - 1 = 5 - 1$. Two errors made the “proof” look like it reached the goal. Correct: $\sum_{i=0}^k F_i + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+3} - 1$.

Spot the error 5 (the largest element). Inserting 9 between 1 and 2 gives 192436587: the ascent 12 is destroyed, and the new pairs are the ascent 19 and the descent 92. The number of ascents stays 4. Inserting 9 inside an ascent or at the start does not add an ascent: 5 locations give four ascents and 4 give five.

Spot the error 6 (odd length). For odd n the middle letter is not mirrored by any other letter but can still be chosen freely, so there are $\lceil n/2 \rceil = \frac{n+1}{2}$ free letters. The “solution” gives $26^{1/2}$ for $n = 1$, which is not an integer. Correct: $26^{\lceil n/2 \rceil}$.