Tutorial 3: Assignment 1, Questions 1 to 7

Books on shelves, round tables, card hands, the multinomial theorem, and inequalities with slack variables

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Count arrangements of distinct objects on shelves by **lining them up and then cutting the row**, with and without the condition "at least one on each shelf".
2. Count seatings at a **round table** with $(n-1)!$, and combine this with the gap method (a circle of m people has m gaps, not $m+1$).
3. Count card hands by choosing the **ranks first and then the suits**, and decide correctly between $13 \cdot 12$ (ranks with different roles) and $\binom{13}{2}$ (ranks with the same role).
4. Count strings with a prescribed number of each symbol with the **multinomial coefficient** $\binom{n}{n_1,\ldots,n_k}$, and split "at least" and "weight" conditions into exact cases.
5. Read off coefficients with the **multinomial theorem**, including numerical coefficients, signs and the constant term, and evaluate sums by **plugging values into the binomial theorem**.
6. Count integer solutions of an **inequality** with a slack variable, and handle **negative lower bounds** by shifting the variables.

Sections 1.1 to 1.7 review the tools with a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 solves Questions 1 to 7 of Assignment 1 in full, with the alternative methods that appeared in student solutions. Section 3 collects the classic mistakes, "spot the error" puzzles, a checklist and extra exercises. Tutorials 1 and 2 (sum and product rules, $P(n,r)$, $\binom{n}{r}$, the gap method, stars and bars) are assumed.

Contents

1 The toolbox

Questions 1 to 7 of Assignment 1 each need one idea that goes a little beyond Tutorials 1 and 2. As before, every tool is explained the same way: the idea in plain words, a small example, the precise statement, a bigger example, and a recap. The examples in this section are *not* the assignment questions; those are solved in Section 2.

1.1 Books on shelves: line them up, then cut

The intuition

Placing distinct books on shelves, where the order on each shelf matters, looks like a two-step choice (which books go where, and in what order). There is a much faster way to see it: **read the shelves one after the other**. Shelf 1 from left to right, then shelf 2 from left to right, gives **one long row of books**, together with the place where shelf 1 ends. So a placement is the same thing as a *row of all the books* plus a *cut* in the row.

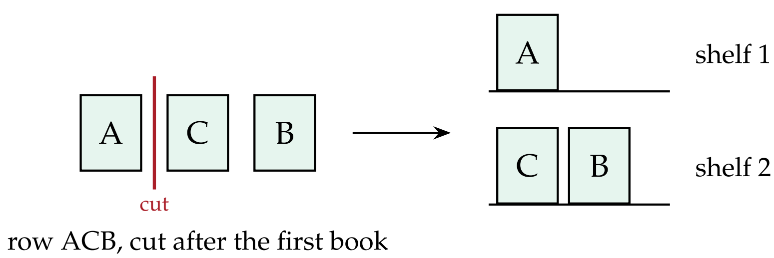


Figure 1. The row ACB with a cut after the first book is the placement "A on shelf 1; C then B on shelf 2". Reading the shelves in order gives the row back, so no placement is counted twice.

Simple example: three books, two shelves

Place the books A, B, C on two shelves with at least one book on each shelf. Line the books up: $3! = 6$ rows. Each row has 2 gaps between consecutive books, and a cut in either gap leaves both shelves non-empty. By the rule of product there are $6 \cdot 2 = 12$ placements.
Check by listing. One book on shelf 1 and two on shelf 2: 3 choices for the single book, 2 orders for the pair: 6. Two books on shelf 1 and one on shelf 2: also 6. Total 12. ✓

Distinct objects on labelled shelves

Let $n \geq 1$ distinct objects be placed on k labelled shelves, where the left-to-right order on each shelf matters.

(i) **Every shelf non-empty.** Arrange the objects in a row ($n!$ ways) and choose $k-1$ of the $n-1$ gaps *between* consecutive objects for the cuts:

$$n!\binom{n-1}{k-1}.$$

(ii) **Empty shelves allowed.** Now two cuts may be in the same place, or at an end of the row. Treat the $k-1$ cuts as identical dividers and arrange the n distinct objects together with them:

$$\frac{(n+k-1)!}{(k-1)!}.$$

Both counts work because reading shelf 1, then shelf 2, ..., then shelf k recovers the row and the positions of the cuts: each placement corresponds to exactly one row with cuts.

Comprehensive example: six books on three shelves

Six different books are placed on three labelled shelves. (a) How many placements if shelves may be empty? (b) How many if every shelf gets at least one book?

Solution. (a) Arrange 6 books and 2 identical dividers: $\dfrac{8!}{2!} = 20\,160$. (b) Arrange the books ($6!$) and choose 2 of the 5 inner gaps for the cuts: $6!\binom{5}{2} = 720 \cdot 10 = 7200$.
Check (b) with inclusion-exclusion. From the 20 160 placements of (a), remove those with an empty shelf. If a given shelf is empty, the books sit on the other two shelves in $\frac{7!}{1!} = 5040$ ways; there are 3 choices of the empty shelf. If two given shelves are empty, all books are on the third shelf: $6! = 720$ ways, 3 choices. So $20\,160 - 3 \cdot 5040 + 3 \cdot 720 = 7200$. ✓

Do not forget the order on the shelves

"Each book chooses a shelf" gives 2^n (or k^n) and only says *which* books are on which shelf. The books on a shelf can still be arranged among themselves, and those arrangements are different placements. Choosing the subsets is only half of the count.

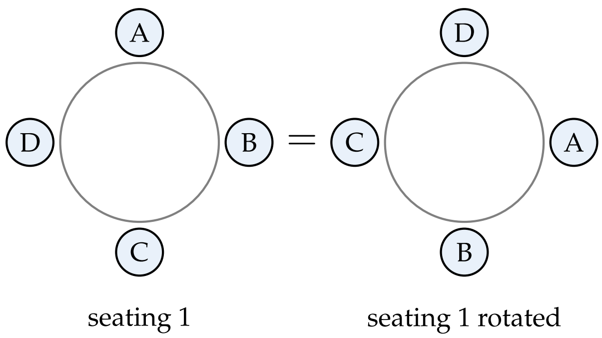
Shelves in one breath

- Placement on shelves = a row of all the objects + cuts. Non-empty shelves: $n!\binom{n-1}{k-1}$. Empty allowed: $\frac{(n+k-1)!}{(k-1)!}$.
- For two non-empty shelves this is simply $n!\,(n-1)$.

1.2 Round tables: fix one person

The intuition

At a round table, what matters is **who sits next to whom**, not which chair someone took: if everybody moves one seat to the left, nothing has changed. The easy way to remove this rotation is to **seat one person anywhere and use them as the reference point**. Everybody else is then placed relative to that person, exactly as in a row.



Same neighbours for everyone:
4 rotations give 1 seating.

4! rows $\rightarrow \dfrac{4!}{4} = 3! = 6$ seatings.

Figure 2. Rotating everybody gives the same circular seating. Each seating corresponds to n rows (cut the circle just before any of the n people), so there are $n!/n = (n-1)!$ seatings.

Simple example: four friends at a round table

Seat A anywhere. Going clockwise from A, the other three people can be arranged in $3! = 6$ ways, and different clockwise orders really are different seatings (someone has a different neighbour). So there are 6 seatings, compared with $4! = 24$ in a row. Each seating has been counted 4 times in the row count, once for each person who could sit "first".

Circular arrangements and the circular gap method

(i) The number of ways to seat n distinct people around a round table, where two seatings are the same if one is a rotation of the other, is $\dfrac{n!}{n} = (n-1)!$.

(ii) If the seats are **numbered** (rotations count as different), the number is $n!$, as in a row.

(iii) **Gap method on a circle.** $m \geq 2$ people seated around a table create exactly m gaps (one between each pair of neighbours), because the gap "after the last person" and the gap "before the first person" are the same gap. In a row, m people create $m+1$ gaps.

Comprehensive example: study group around a table

Five CS students and three Math students (all different) sit at a round table with 8 seats. In how many seatings are no two Math students next to each other?

Solution. First seat the 5 CS students around the table: $(5-1)! = 4! = 24$ ways. They create 5 gaps. Two Math students are neighbours exactly when they are in the same gap, so we choose 3 of the 5 gaps, $\binom{5}{3} = 10$ ways, and place the 3 Math students in them, $3! = 6$ ways. By the rule of product: $24 \cdot 10 \cdot 6 = 1440$.

Check. Choosing 3 gaps and then ordering the Math students is the same as placing them in 3 distinct gaps one by one: $P(5,3) = 60$, and $24 \cdot 60 = 1440$. ✓

Two classic slips at a round table

- Using $n!$ for the first group: this counts rotations as different, but the question treats them as the same. Once one group is seated around the table, the rotation is fixed, so the second group is placed with ordinary (non-circular) counting.
- Using the *row* number of gaps, $m+1$. On a circle the two end gaps are the same gap, and two people placed "at both ends" would sit next to each other.

Round tables in one breath

- n people around a table: $(n-1)!$. Numbered seats: $n!$.
- Keep a group apart: seat the others around the table first ($(m-1)!$), then choose among the m gaps.

1.3 Card hands: ranks first, then suits

The intuition

A standard deck has 52 cards: 13 **ranks** (A, 2, 3, ..., 10, J, Q, K) in each of 4 **suits** (♠, ♡, ♢, ♣). A card is a pair (rank, suit). Most hands are described by the *pattern of ranks*: a full house is "three of one rank and two of another". So count in two stages: **choose the ranks** that play each role in the pattern, **then choose the suits** (which cards of each chosen rank).

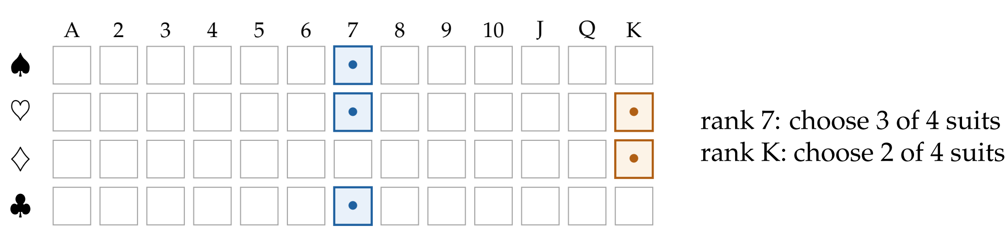


Figure 3. The deck as a 4×13 grid (suits $\times$ ranks). The highlighted hand is a full house: three 7's and two kings. First the ranks are chosen (columns), then the suits inside each chosen column.

Simple example: two-card hands

A two-card hand is either a *pair* (both cards of one rank) or two cards of *different ranks*.

- Pair: choose the rank (13), then 2 of its 4 suits ($\binom{4}{2} = 6$): $13 \cdot 6 = 78$.
- Different ranks: the two ranks play the *same* role, so choose them together: $\binom{13}{2} = 78$; then one suit for each: $4 \cdot 4$. Total $78 \cdot 16 = 1248$.

Check: $78 + 1248 = 1326 = \binom{52}{2}$, the number of all two-card hands. ✓

Counting a hand by its rank pattern

Suppose a hand type is described by a pattern of ranks, such as $3+2$ (full house) or $2+1+1+1$ (one pair), where different numbers mean different ranks. Then

$$\#\text{hands} \;=\; \Big(\#\text{ways to choose the ranks for the roles}\Big) \times \prod_{\text{chosen ranks}} \binom{4}{\#\text{cards of that rank}}.$$

- Ranks that play **different roles** (the rank of the triple and the rank of the pair) are chosen **in order**: $13 \cdot 12$.
- Ranks that play **the same role** (the ranks of two pairs, or the ranks of the single cards) are chosen **as a set**: $\binom{13}{2}$, $\binom{12}{3}$, ...
- Hand names mean the **exact** pattern: "three of a kind" is $3+1+1$ with the two single cards of different ranks, so it contains neither a full house nor four of a kind.

Comprehensive example: one pair

How many five-card hands contain *exactly* one pair (pattern $2+1+1+1$)?

Solution. Rank of the pair: 13 ways. Ranks of the three single cards: they play the same role, and they must be different from each other and from the pair's rank, so $\binom{12}{3} = 220$ ways. Suits: $\binom{4}{2} = 6$ for the pair and 4 for each single card. By the rule of product:

$$13 \cdot \binom{12}{3} \cdot \binom{4}{2} \cdot 4^3 \;=\; 13 \cdot 220 \cdot 6 \cdot 64 \;=\; 1\,098\,240.$$

Ordered or unordered ranks?

The test is: *if I swap the two ranks, do I get a different hand?* In a full house, "three 7's and two K's" is not "three K's and two 7's", so order matters ($13 \cdot 12$). With two pairs, "7's and K's" is the same hand as "K's and 7's", so order does not matter ($\binom{13}{2}$). Getting this wrong changes the answer by a factor of exactly 2 (or 3! for three ranks).

Card hands in one breath

- Write the rank pattern, choose the ranks (ordered for different roles, a set for equal roles), then the suits.
- "Exactly" is the default: make sure the extra cards cannot upgrade the hand.

1.4 Strings with a prescribed composition

The intuition

A *ternary string* is a string of the symbols $0, 1, 2$. If we know **how many** of each symbol the string contains, counting the strings is the BANANAS problem of Tutorial 1: arrange a multiset. The new part is what to do with conditions like "at least eight 1's" or "weight 4" (the *weight* of a string is the sum of its digits): **list the possible exact compositions**, count each one, and add.

Simple example: ternary strings of length 4

How many strings of length 4 have two 0's, one 1 and one 2? Choose the positions of the 0's ($\binom{4}{2} = 6$), then the position of the 1 among the remaining two ($\binom{2}{1} = 2$); the 2 goes in the last place. So $6 \cdot 2 = 12$, which is also $\dfrac{4!}{2!\,1!\,1!} = 12$.

Multinomial coefficients

Let $n = n_1 + n_2 + \cdots + n_k$ with all $n_i \geq 0$. The number of strings of length n with exactly n_i copies of symbol i (for $i = 1, \ldots, k$) is the **multinomial coefficient**

$$\binom{n}{n_1,\, n_2,\, \ldots,\, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!} = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_k}{n_k}.$$

The product form says: choose the positions of symbol 1, then of symbol 2 among the remaining positions, and so on. For a condition that allows several compositions ("at least", "weight w"), sum the multinomial coefficient over all compositions that satisfy the condition; different compositions give different strings.

Comprehensive example: weight 3, length 6

How many ternary strings of length 6 have weight 3?
Solution. Let a be the number of 1's and b the number of 2's. The weight is $a + 2b = 3$, so $(a, b) \in \{(3, 0), (1, 1)\}$, and the remaining positions are 0's.

- $(3, 0)$: three 1's and three 0's: $\binom{6}{3} = 20$.
- $(1, 1)$: one 1, one 2 and four 0's: $\dfrac{6!}{1!\,1!\,4!} = 30$.

By the rule of sum, $20 + 30 = 50$ strings.
Check with stars and bars. A string of weight 3 is a solution of $x_1 + \cdots + x_6 = 3$ with each $x_i \in \{0, 1, 2\}$. There are $\binom{6+3-1}{3} = 56$ non-negative solutions, and exactly 6 of them use a digit 3 (not allowed). $56 - 6 = 50$. ✓

Compositions in one breath

- Exact composition $(n_1, \ldots, n_k)$: $\dfrac{n!}{n_1! \cdots n_k!}$.
- "At least", "at most", "weight": list the exact compositions, one multinomial coefficient each, then add.

1.5 The binomial and multinomial theorems

The intuition

Expanding $(a+b+c)^4 = (a+b+c)(a+b+c)(a+b+c)(a+b+c)$ means: **from each of the 4 brackets pick one term** and multiply. Every choice is a string of length 4 over $\{a, b, c\}$. The monomial a^2bc appears once for every string with two a's, one b and one c. So **the coefficient is a multinomial coefficient**, exactly the count of Section 1.4.

$$\begin{array}{ccccc} (a+b+c) & (a+b+c) & (a+b+c) & (a+b+c) & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ a & b & a & c & \Rightarrow \text{the string } a\,b\,a\,c \text{ gives } a^2bc \end{array}$$

Figure 4. One term is picked from each bracket. The 12 strings with two a's, one b and one c each produce a^2bc, so the coefficient of a^2bc in $(a+b+c)^4$ is $\frac{4!}{2!\,1!\,1!} = 12$.

Simple example: numbers in front of the variables

The coefficient of x^2y in $(2x-3y)^3$: the monomial comes from choosing the term $2x$ from 2 of the 3 brackets and $-3y$ from the other one. There are $\binom{3}{2} = 3$ such choices, and each contributes $(2x)^2(-3y) = -12\,x^2y$. So the coefficient is $3 \cdot 4 \cdot (-3) = -36$.

Binomial and multinomial theorems

For every integer $n \geq 0$:

(i) **Binomial theorem.** $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

(ii) **Multinomial theorem.** $(x_1 + x_2 + \cdots + x_t)^n = \sum_{\substack{n_1+\cdots+n_t=n \\ n_i \geq 0}} \binom{n}{n_1, \ldots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$.

(iii) **Terms with numbers.** If the brackets contain terms c_1x_1, c_2x_2, … and a constant c_0, write each term as one "block" $X_i = c_ix_i$. The coefficient of $x_1^{n_1} \cdots x_t^{n_t}$ in $(c_1x_1 + \cdots + c_tx_t + c_0)^n$ is

$$\binom{n}{n_1, \ldots, n_t,\ n_0} c_1^{n_1} \cdots c_t^{n_t}\, c_0^{n_0}, \qquad \text{where } n_0 = n - (n_1 + \cdots + n_t).$$

A variable that does not appear in the monomial has exponent 0 (and contributes $0! = 1$ and $c^0 = 1$).

Comprehensive example: a constant term and a sign

Find the coefficient of x^2y^3 in $(x-2y+3)^7$.
Solution. The blocks are x, $-2y$ and 3. The exponents are 2 for x, 3 for y, and the constant gets the rest: $n_0 = 7-2-3 = 2$. So the coefficient is

$$\binom{7}{2,3,2} \cdot 1^2 \cdot (-2)^3 \cdot 3^2 \;=\; \frac{7!}{2!\,3!\,2!} \cdot (-8) \cdot 9 \;=\; 210 \cdot (-72) \;=\; -15\,120.$$

The three things that get lost

1. **The constant's exponent.** The exponents must add up to n, the power of the bracket, not to the degree of the monomial. The constant takes whatever is left.
2. **The numbers in the blocks.** Each number is raised to its block's exponent: $(-2y)^3$ gives $(-2)^3$, not -2.
3. **The sign.** An odd power of a negative number is negative; an even power is positive.

Multinomial theorem in one breath

- Coefficient = (number of ways to pick the blocks) × (the numbers in the blocks, to their powers).
- Exponents add up to n; the constant term is a block too.

1.6 Sums you can read off the binomial theorem

The intuition

The binomial theorem $(1+x)^n = \sum_k \binom{n}{k} x^k$ is true **for every number** x. Plugging in a well-chosen value turns the right-hand side into a sum we want and the left-hand side into a simple number. The only work is to **recognise** $\binom{n}{k}$ **inside the sum**: if you see $\frac{1}{k!\,(n-k)!}$, multiply and divide by $n!$.

Simple example: $x = 1$ and $x = -1$

With $x = 1$: $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$. With $x = -1$: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0$ for $n \geq 1$. For $n = 3$: $1+3+3+1 = 8 = 2^3$ and $1-3+3-1 = 0$. ✓

Evaluating sums with the binomial theorem

For every integer $n \geq 0$ and every number x,

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n, \qquad \text{and} \qquad \frac{1}{k!\,(n-k)!} = \frac{1}{n!}\binom{n}{k}.$$

In particular $\sum_k \binom{n}{k} = 2^n$, and $\sum_k (-1)^k \binom{n}{k} = 0$ for $n \geq 1$ (for $n = 0$ the sum is 1, because $(1-1)^0 = 0^0 = 1$).

Comprehensive example: a factor 2^i

Evaluate $S = \sum_{i=0}^{n} \frac{2^i}{i!\,(n-i)!}$ for $n \geq 1$.

Solution. Multiply and divide by $n!$, then use the binomial theorem with $x = 2$:

$$S = \frac{1}{n!}\sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!}\,2^i = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}2^i = \frac{(1+2)^n}{n!} = \frac{3^n}{n!}.$$

Check for $n = 2$: $\frac{1}{0!\,2!} + \frac{2}{1!\,1!} + \frac{4}{2!\,0!} = \frac{1}{2} + 2 + 2 = \frac{9}{2} = \frac{3^2}{2!}$. ✓

Binomial sums in one breath

- Spot $\frac{1}{k!(n-k)!}$, pull out $\frac{1}{n!}$ to create $\binom{n}{k}$, then choose x in $(1+x)^n$.
- Always check the formula for $n = 1$ or $n = 2$.

1.7 Inequalities and lower bounds: slack variables

The intuition

Stars and bars (Tutorial 2) counts solutions of an *equation*. For an inequality like $x_1 + x_2 + x_3 \leq 10$, add one more variable, the **slack** s, that collects **what is left over**: $x_1 + x_2 + x_3 + s = 10$. Every solution of the inequality gives exactly one solution of the equation (with $s = 10 - (x_1 + x_2 + x_3) \geq 0$), and back. For a lower bound like $x_i \geq -3$, **shift the variable** so that it starts at 0.

$$\underbrace{\star\ \star\ \star}_{x_1=3}\;\Big|\;\underbrace{\star\ \star}_{x_2=2}\;\Big|\;\underbrace{\star\ \star\ \star}_{x_3=3}\;\Big|\;\underbrace{\star\ \star}_{s=2}$$

$3 + 2 + 3 = 8 \leq 10$,
the slack takes the other 2

Figure 5. The solution $(3, 2, 3)$ of $x_1 + x_2 + x_3 \leq 10$ becomes the solution $(3, 2, 3, 2)$ of $x_1 + x_2 + x_3 + s = 10$: the last part of the stars-and-bars picture belongs to the slack variable.

Simple example: $x_1 + x_2 \leq 2$

Listing the non-negative solutions: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(2, 0)$, $(1, 1)$, $(0, 2)$, so 6 solutions. With a slack: $x_1 + x_2 + s = 2$, $s \geq 0$, has $\binom{3+2-1}{2} = \binom{4}{2} = 6$ solutions. ✓

Inequalities and shifted lower bounds

Let $n \geq 1$ and $r \geq 0$ be integers, and let all x_i be integers.

(i) **Weak inequality.** The number of solutions of $x_1 + \cdots + x_n \leq r$ with all $x_i \geq 0$ equals the number of solutions of $x_1 + \cdots + x_n + s = r$ with all variables ≥ 0, namely

$$\binom{(n+1) + r - 1}{r} = \binom{n+r}{r}.$$

(ii) **Strict inequality.** For integers, $x_1 + \cdots + x_n < r$ is the same as $x_1 + \cdots + x_n \leq r - 1$.

Equivalently, use a slack $s \geq 1$ with $x_1 + \cdots + x_n + s = r$. Both give

$$\binom{n+r-1}{r-1}.$$

(iii) **Lower bounds, also negative ones.** If $x_i \geq b_i$, put $y_i = x_i - b_i \geq 0$. Then $\sum x_i = \sum y_i + \sum b_i$, so the right-hand side changes from r to $r - \sum b_i$. A *negative* lower bound b_i *increases* the right-hand side.

Comprehensive example: mixed lower bounds

How many integer solutions does $x_1 + x_2 + x_3 \leq 10$ have with $x_1 \geq 2$, $x_2 \geq -1$, $x_3 \geq 0$?
Solution. Put $y_1 = x_1 - 2 \geq 0$ and $y_2 = x_2 + 1 \geq 0$. Then $x_1 + x_2 + x_3 = y_1 + y_2 + x_3 + 1$, and the condition becomes $y_1 + y_2 + x_3 \leq 9$. With a slack $s \geq 0$: $y_1 + y_2 + x_3 + s = 9$, which has $\binom{4+9-1}{9} = \binom{12}{3} = 220$ solutions.
Check. Count by the exact value $t = y_1 + y_2 + x_3 \in \{0, \ldots, 9\}$: $\sum_{t=0}^{9} \binom{t+2}{2} = 1 + 3 + 6 + \cdots + 55 = 220$. ✓

Strict inequality and the direction of the shift

- For "$< r$" the slack must be at least 1 (or use "$\leq r - 1$"). A slack $s \geq 0$ with sum r also counts the solutions with sum exactly r.
- For $x_i \geq -3$ the substitution is $y_i = x_i + 3$ (so that $y_i \geq 0$). Sanity check: allowing negative values gives *more* solutions, so the answer must go *up*.

Slack variables in one breath

- $\leq r$: add $s \geq 0$. $< r$: use $\leq r - 1$, or $s \geq 1$.
- Lower bound b_i: substitute $y_i = x_i - b_i$ and replace r by $r - \sum b_i$; then stars and bars.

The toolbox at a glance

Situation	Count / method	Signal words
n distinct objects on k shelves, none empty	$n!\binom{n-1}{k-1}$	"shelves", "at least one on each"
Same, shelves may be empty	$\frac{(n+k-1)!}{(k-1)!}$	"shelves", no condition
n people around a table	$(n-1)!$	"circular", "round table"
Keep a group apart around a table	$(m-1)!$, then choose among m gaps	"no two … next to each other"
Card hands	ranks (ordered or set), then suits	"flush", "pair", "full house"
Strings with n_i copies of symbol i	$\frac{n!}{n_1! \cdots n_k!}$	"four 0's, three 1's", "weight"
Coefficient in $(c_1 x_1 + \cdots + c_0)^n$	$\binom{n}{n_1, \ldots, n_0} c_1^{n_1} \cdots c_0^{n_0}$	"coefficient of", "expansion"
Sums with $\frac{1}{i!(n-i)!}$	multiply by $\frac{n!}{n!}$, then $(1+x)^n$	"for any positive integer n determine"
$\sum x_i \leq r$ or $< r$, $x_i \geq b_i$	shift, add a slack, stars and bars	"inequality", "$x_i \geq -3$"

2 The assignment problems, worked in full

2.1 How to write the solutions

All seven questions are written with the counting template from Tutorial 1: **Statement** $\to$ **Solution** (set-up, then a step-by-step count naming the rule used at each step) $\to$ **Answer**, with an optional *Check*. Question 6 is a computation rather than a count, but the same skeleton works: state what you evaluate, justify each equality, box the result. The assignment asks you to **show all the steps**: a correct number with no argument is not a solution.

Writing template for a counting solution

Statement. *[Restate the question precisely: what objects, what makes one "valid", what counts as "the same" (rotations? order on a shelf?).]*
Solution. *Set-up. [Say what you are counting and settle the questions: does order matter? is repetition allowed? are some objects identical? are the containers or seats labelled?]*
Count. [Go step by step. After each factor name the rule. If a step has cases, list them, explain why the list is complete, and add.]
Answer. *[Give the expression and its numerical value.]*
Check (optional but recommended). [A small case you can list, or a second method giving the same number.]

Phrasebook: new sentences for this tutorial

Where	What to write
Shelves	"Reading shelf 1 and then shelf 2 from left to right gives a row of all the books; conversely a row and a cut determine the placement."
Round table	"Since rotations give the same seating, we fix the position of one woman and arrange the others relative to her: 8! ways."
Circular gaps	"The 9 women create exactly 9 gaps around the table; no two men are adjacent if and only if each gap receives at most one man."
Card hands	"We first choose the ranks and then the suits." "The two pair ranks play the same role, so we choose them as a set: $\binom{13}{2}$."
Compositions	"Let a be the number of 1's and b the number of 2's; the weight is $a + 2b$."
Multinomial	"By the multinomial theorem with the blocks $2w, -x, 3y, z, -2$, the coefficient is . . . "
Slack	"Since the x_i are integers, $\sum x_i < 40$ is equivalent to $\sum x_i \leq 39$. Let $x_6 = 39 - \sum x_i \geq 0$."
Shift	"Put $y_i = x_i + 3$; then $y_i \geq 0$ and $\sum y_i = \sum x_i + 15$."

2.2 Problem 1: Pamela's bookshelves

Problem 1 (Exercises 1.1 and 1.2, #10)

Pamela has 15 different books. In how many ways can she place her books on two shelves so that there is at least one book on each shelf? (Consider the books in each arrangement to be stacked one next to the other, with the first book on each shelf at the left of the shelf.)

How to think about it. The books are different, and the order on each shelf matters ("the first book on each shelf at the left"). The two shelves are different too (top and bottom). This is Section 1.1 with $n = 15$ and $k = 2$.

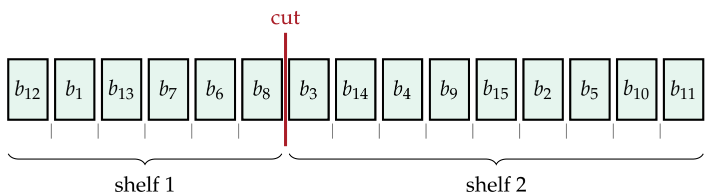


Figure 6. One of the 15! rows of the books $b_1, \ldots, b_{15}$, with the cut after the sixth book. The small ticks are the 14 gaps where the cut may go; a cut at either end would leave a shelf empty.

Model solution to Problem 1

Solution (what you write)	Commentary (what it does)
Statement. Count the placements of 15 distinct books on two (distinguishable) shelves, where the left-to-right order on each shelf matters and each shelf holds at least one book.	*Say explicitly that order on a shelf matters: that is what the sentence in brackets in the question means.*
Solution. Reading shelf 1 from left to right and then shelf 2 from left to right turns a placement into a row of all 15 books together with the position where shelf 1 ends. Conversely, a row and a cut position determine the placement. So we count rows with a cut. The books can be arranged in a row in 15! ways. The cut must go in one of the 14 gaps between two consecutive books (a cut at an end would leave a shelf empty), so there are 14 choices for each row. By the rule of product the number of placements is $15! \cdot 14$.	*The "conversely" sentence is the proof that nothing is counted twice or missed (a bijection).*
Answer. $\boxed{14 \cdot 15! = 18\,307\,441\,152\,000}$.	
Check. Split by j, the number of books on shelf 1, $1 \le j \le 14$. Choose them ($\binom{15}{j}$), order them ($j!$) and order the rest on shelf 2 ($(15-j)!$): $\binom{15}{j}\, j!\,(15-j)! = 15!$ for every j. Adding over the 14 values of j gives $14 \cdot 15!$. ✓	*This case-split method was the most common in the submitted solutions. It is correct, just longer.*

A third way, with a divider

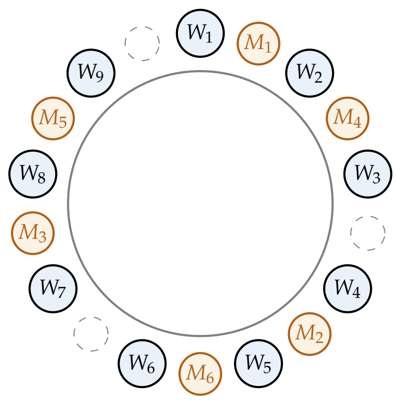
Treat the end of shelf 1 as a divider $|$ and arrange the 15 books and the divider: 16! rows. The divider may not be first or last (empty shelf): 15! rows each. So $16! - 2 \cdot 15! = 15!\,(16-2) = 14 \cdot 15!$. Three methods, one number.

2.3 Problem 2: the committee at a round table

Problem 2 (Exercises 1.1 and 1.2, #38)

A committee of 15 (nine women and six men) is to be seated at a circular table (with 15 seats). In how many ways can the seats be assigned so that no two men are seated next to each other?

How to think about it. Two tools from Section 1.2: seat the group that has no restriction (the women) around the table first, then place the men in the gaps between the women, at most one man per gap. With 9 women on a circle there are 9 gaps.



women: 8! circular orders
9 gaps, choose 6: $\binom{9}{6}$
men in the chosen gaps: 6!
dashed: gaps left empty

Figure 7. The women create 9 gaps around the table (between W_9 and W_1 too). Six of them receive one man each; the other three are empty, which means those two women sit next to each other.

Model solution to Problem 2

Solution (what you write)	**Commentary (what it does)**
Statement. Count the seatings of 9 distinct women and 6 distinct men around a round table with 15 seats, where seatings that differ by a rotation are the same, such that no two men are adjacent.	*State the convention for rotations; the textbook uses "rotations are the same" for circular tables.*
Solution. *Step 1: the women.* Since rotations give the same seating, fix the position of one woman and arrange the other 8 relative to her: 8! ways. *Step 2: choose the gaps.* Going around the table, the 9 women create exactly 9 gaps, one between each pair of neighbouring women. Two men are adjacent if and only if they are in the same gap, so each gap receives at most one man. Since there are 15 seats and 15 people, the men fill 6 of the gaps, chosen in $\binom{9}{6}$ ways. *Step 3: the men.* The 6 distinct men are assigned to the 6 chosen gaps in 6! ways. By the rule of product the number of seatings is $8! \cdot \binom{9}{6} \cdot 6!$.	*The women are seated first because they have no restriction; the men are then kept apart by the women. After Step 1 the table has a fixed orientation, so Steps 2 and 3 are ordinary (non-circular) counting.*
Answer. $8!\binom{9}{6}\,6! = \dfrac{8!\,9!}{3!}$ $\boxed{= 40\,320 \cdot 84 \cdot 720 = 2\,438\,553\,600}$.	*Note that $\binom{9}{6}\,6! = P(9,6)$: choosing 6 gaps and ordering the men is placing the men one by one.*

What if the seats are numbered?

If the 15 seats are considered different (a rotation gives a new assignment), then every circular seating corresponds to 15 seat assignments, and the answer is $15 \cdot 8!\binom{9}{6}\, 6! = 36\,578\,304\,000$. One student solution added this remark; it is a good habit to state which reading you use. The textbook answer uses the circular reading.

2.4 Problem 3: five-card hands

Problem 3 (Exercises 1.3, #8)

In how many ways can a gambler draw five cards from a standard deck and get (a) a flush (five cards of the same suit)? (b) four aces? (c) four of a kind? (d) three aces and two jacks? (e) three aces and a pair? (f) a full house (three of a kind and a pair)? (g) three of a kind? (h) two pairs?

How to think about it. A hand is a set of 5 cards (order of drawing does not matter). Every part is Section 1.3: write the rank pattern, choose the ranks, then choose the suits. The pattern of each part is in Table 1; the only delicate decisions are ordered versus unordered ranks, and making sure that parts (g) and (h) mean the *exact* pattern.

Part	Hand	Rank pattern	Count
(a)	flush	one suit, 5 ranks	$\binom{4}{1}\binom{13}{5} = 5148$
(b)	four aces	$4+1$, the 4 are aces	$\binom{4}{4} \cdot 48 = 48$
(c)	four of a kind	$4+1$	$13 \cdot \binom{4}{4} \cdot 48 = 624$
(d)	three aces, two jacks	$3+2$, both ranks fixed	$\binom{4}{3}\binom{4}{2} = 24$
(e)	three aces and a pair	$3+2$, triple rank fixed	$\binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 288$
(f)	full house	$3+2$	$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3744$
(g)	three of a kind	$3+1+1$	$13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2 = 54\,912$
(h)	two pairs	$2+2+1$	$\binom{13}{2}\binom{4}{2}^2 \cdot 44 = 123\,552$

Table 1. Problem 3 at a glance. Different numbers in a pattern mean different ranks.

Model solution to Problem 3(a) to (c)

Solution (what you write)	Commentary (what it does)
Statement. A hand is a 5-element subset of the 52 cards (13 ranks, 4 suits). Count the hands of each type.	*One statement for all parts; then one short argument per part.*
(a) Flush. Choose the suit: $\binom{4}{1} = 4$ ways. Then choose 5 of the 13 cards of that suit: $\binom{13}{5} = 1287$. By the rule of product: $\boxed{4\binom{13}{5} = 5148}$.	*The question defines a flush as five cards of one suit, so straight flushes are included.*
(b) Four aces. All four aces are in the hand ($\binom{4}{4} = 1$ way), and the fifth card is any of the $52 - 4 = 48$ other cards: $\boxed{48}$.	

(c) Four of a kind. Choose the rank of the four: 13 ways; take all four cards of that rank; the fifth card is any of the 48 other cards. By the rule of product: $\boxed{13 \cdot 48 = 624}$.

Part (b) is the special case "the rank is ace". The fifth card can never make a different hand of this type.

Model solution to Problem 3(d) to (f)

Solution (what you write)	Commentary (what it does)
(d) Three aces and two jacks. Choose 3 of the 4 aces and 2 of the 4 jacks: $\boxed{\binom{4}{3}\binom{4}{2} = 4 \cdot 6 = 24}$.	*Both ranks are given, so only the suits are chosen.*
(e) Three aces and a pair. Choose 3 aces: $\binom{4}{3} = 4$. The pair has a rank different from ace (only one ace is left), so 12 choices of rank and $\binom{4}{2} = 6$ choices of suits. By the rule of product: $\boxed{4 \cdot 12 \cdot 6 = 288}$.	*Always say why the pair's rank is not the triple's rank.*
(f) Full house. Choose the rank of the triple (13), its suits ($\binom{4}{3} = 4$), the rank of the pair among the other 12 ranks (12), and its suits ($\binom{4}{2} = 6$). By the rule of product: $\boxed{13 \cdot 4 \cdot 12 \cdot 6 = 3744}$.	*The two ranks play different roles, so they are chosen in order:* $13 \cdot 12$, *not* $\binom{13}{2}$.

Model solution to Problem 3(g) and (h)

Solution (what you write)	Commentary (what it does)
(g) Three of a kind. We count hands with pattern $3 + 1 + 1$: three cards of one rank and two cards of two further, different ranks (otherwise the hand would be a full house or four of a kind). Rank of the triple: 13; its suits: $\binom{4}{3} = 4$. Ranks of the two single cards: they play the same role, so choose 2 of the remaining 12 ranks as a set: $\binom{12}{2} = 66$. One suit for each single card: $4 \cdot 4$. By the rule of product: $\boxed{13 \cdot 4 \cdot 66 \cdot 16 = 54\,912}$.	*Interpret the hand as the* exact *pattern and say so in one sentence.*
Check (complement). Triple, then any 2 of the 48 cards of other ranks: $13 \cdot 4 \cdot \binom{48}{2} = 58\,656$. This also contains the full houses (the two extra cards form a pair), so $58\,656 - 3744 = 54\,912$. ✓	*This was another student's method: count a bigger set, then remove what does not belong.*
(h) Two pairs. Pattern $2 + 2 + 1$. Ranks of the two pairs: same role, so $\binom{13}{2} = 78$. Suits of each pair: $\binom{4}{2} \cdot \binom{4}{2} = 36$. The fifth card must have a third rank, so it is one of the $52 - 8 = 44$ cards whose rank is neither pair rank. By the rule of product: $\boxed{78 \cdot 36 \cdot 44 = 123\,552}$.	*Equivalently: third rank 11 ways, suit 4 ways,* $11 \cdot 4 = 44$.

A global check

The six exact rank patterns 1+1+1+1+1, 2+1+1+1, 2+2+1, 3+1+1, 3+2, 4+1 cover every hand exactly once, so their counts must add up to $\binom{52}{5} = 2\,598\,960$. With (c), (f), (g), (h) and the "one pair" example of Section 1.3, only the "five different ranks" count is missing. Exercise E3 asks you to finish this check.

2.5 Problem 4: ternary strings of length 10

Problem 4 (Exercises 1.3, #18)

For the strings of length 10 in Example 1.24 from the textbook, how many have (a) four 0's, three 1's, and three 2's; (b) at least eight 1's; (c) weight 4?

How to think about it. The strings of Example 1.24 are the *ternary* strings: strings of length 10 over $\{0, 1, 2\}$, and the weight of a string is the sum of its digits. Part (a) is one multinomial coefficient; parts (b) and (c) list the exact compositions (Section 1.4).

Model solution to Problem 4(a)

Solution (what you write)	Commentary (what it does)
Statement. Count the strings of length 10 with exactly four 0's, three 1's and three 2's.	
Solution. Choose the positions of the 0's ($\binom{10}{4}$), then of the 1's among the other 6 positions ($\binom{6}{3}$); the 2's fill the last 3 positions ($\binom{3}{3}$). By the rule of product the count is $\binom{10}{4}\binom{6}{3}\binom{3}{3} = \frac{10!}{4!\,3!\,3!}$.	*Either form is fine: positions one symbol at a time, or "10! divided by the repeats".*
Answer. $\boxed{\frac{10!}{4!\,3!\,3!} = 210 \cdot 20 \cdot 1 = 4200}$.	

Model solution to Problem 4(b)

Solution (what you write)	Commentary (what it does)
Statement. Count the ternary strings of length 10 with at least eight 1's.	
Solution. Such a string has exactly k 1's for exactly one $k \in \{8, 9, 10\}$. For a fixed k, choose the positions of the 1's ($\binom{10}{k}$), and fill each of the other $10 - k$ positions with 0 or 2 (2^{10-k} ways). $k = 8$: $\binom{10}{8}2^2 = 180$; $k = 9$: $\binom{10}{9}2^1 = 20$; $k = 10$: $\binom{10}{10}2^0 = 1$. The cases are disjoint, so we add them.	*"At least" becomes "exactly k" for each possible k. The other positions must avoid 1, so 2 choices each.*
Answer. $\boxed{\binom{10}{8}2^2 + \binom{10}{9}2 + \binom{10}{10} = 180 + 20 + 1 = 201}$.	

1's (a)	2's (b)	0's	Weight $a+2b$	Number of strings
4	0	6	4	$\binom{10}{4}=210$
2	1	7	4	$\dfrac{10!}{2!\,1!\,7!}=360$
0	2	8	4	$\binom{10}{2}=45$
				total $=615$

Table 2. Problem 4(c): the three compositions of weight 4. Every string of weight 4 is in exactly one row.

Model solution to Problem 4(c)

Solution (what you write)	Commentary (what it does)
Statement. Count the ternary strings of length 10 whose digits add up to 4.	
Solution. Let a be the number of 1's and b the number of 2's; the other $10-a-b$ digits are 0's. The weight is $a+2b=4$ with $a,b \geq 0$, so $b \in \{0,1,2\}$ and $(a,b) \in \{(4,0),(2,1),(0,2)\}$. These three cases are disjoint and cover all strings of weight 4. In each case the count is a multinomial coefficient (Table 2): $\binom{10}{4}=210$, $\frac{10!}{2!\,1!\,7!}=360$, $\binom{10}{2}=45$. By the rule of sum we add them.	*Solve $a+2b=4$ first, and say why the list is complete (here: $b \leq 2$ since $2b \leq 4$).*
Answer. $\boxed{210+360+45=615}$.	
Check. For the middle row: place the 2 (10 ways), then the two 1's among the remaining 9 positions ($\binom{9}{2}=36$): $10 \cdot 36 = 360$. ✓	*Several student solutions computed this row in this way, or as $\binom{10}{3}\cdot 3$.*

2.6 Problem 5: coefficients in multinomial expansions

Problem 5 (Exercises 1.3, #26)

Find the coefficient of $w^2x^2y^2z^2$ in the expansion of (a) $(w+x+y+z+1)^{10}$, (b) $(2w-x+3y+z-2)^{12}$, and (c) $(v+w-2x+y+5z+3)^{12}$.

How to think about it. Section 1.5. Write each bracket as a sum of blocks, give each block its exponent (the constant gets n minus 8, and v in part (c) gets 0), then multiply the multinomial coefficient by the block numbers raised to their exponents. Table 3 organises the bookkeeping.

Part	Blocks	Exponents	Numbers to their powers
(a)	$w,\ x,\ y,\ z,\ 1$	$2,2,2,2,\ 2$	1
(b)	$2w,\ -x,\ 3y,\ z,\ -2$	$2,2,2,2,\ 4$	$2^2(-1)^2 3^2 1^2(-2)^4 = 576$
(c)	$v,\ w,\ -2x,\ y,\ 5z,\ 3$	$0,\ 2,2,2,2,\ 4$	$1^0 1^2(-2)^2 1^2 5^2 3^4 = 8100$

Table 3. Problem 5: in each row the exponents add up to the power of the bracket (10, 12, 12)*.*

Model solution to Problem 5

Solution (what you write)	Commentary (what it does)
(a) The monomial $w^2x^2y^2z^2$ uses 8 of the 10 factors, so the term 1 is chosen from the other 2. By the multinomial theorem the coefficient is $\binom{10}{2,2,2,2,2} = \dfrac{10!}{(2!)^5} = \dfrac{3\,628\,800}{32} = \boxed{113\,400}$.	*$w^2x^2y^2z^2 = w^2x^2y^2z^2 \cdot 1^2$: the hidden 1^2 is where the fifth $2!$ comes from.*
(b) Use the blocks $2w$, $-x$, $3y$, z, -2 with exponents $2,2,2,2$ and $12-8=4$. The term of the expansion is $\binom{12}{2,2,2,2,4}(2w)^2(-x)^2(3y)^2z^2(-2)^4$, so the coefficient is $\dfrac{12!}{(2!)^4\,4!} \cdot 2^2(-1)^2 3^2(-2)^4 = 1\,247\,400 \cdot 576 = \boxed{718\,502\,400}$.	*Write the whole term with the blocks in brackets, then collect the numbers. All powers are even, so the sign is $+$.*
(c) Use the blocks v, w, $-2x$, y, $5z$, 3 with exponents 0 (for v), $2,2,2,2$ and $12-8=4$ (for the constant). The coefficient is $\dfrac{12!}{0!\,(2!)^4\,4!} \cdot (-2)^2\,5^2\,3^4 = 1\,247\,400 \cdot 8100 = \boxed{10\,103\,940\,000}$.	*v does not appear in the monomial, so it gets exponent 0: it contributes $0! = 1$ and $1^0 = 1$, but it is still a block.*

2.7 Problem 6: two binomial sums

Problem 6 (Exercises 1.3, #28)

For any positive integer n determine

$$\text{(a)} \quad \sum_{i=0}^{n} \frac{1}{i!\,(n-i)!} \qquad\qquad \text{(b)} \quad \sum_{i=0}^{n} \frac{(-1)^i}{i!\,(n-i)!}.$$

How to think about it. Section 1.6: the denominators are the denominator of $\binom{n}{i}$. Multiply and divide by $n!$, and then use the binomial theorem with $x = 1$ and $x = -1$.

Model solution to Problem 6

Solution (what you write)	Commentary (what it does)
Statement. Let $n \geq 1$. Evaluate the two sums in closed form.	
(a) Since $n!$ does not depend on i, $$\sum_{i=0}^{n} \frac{1}{i!\,(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i} = \frac{(1+1)^n}{n!},$$ by the binomial theorem $(1+x)^n = \sum_i \binom{n}{i} x^i$ with $x = 1$. Hence the sum is $\boxed{\frac{2^n}{n!}}$.	*Name the theorem and the value of x. Graders want to see where 2^n comes from.*
(b) In the same way, $$\sum_{i=0}^{n} \frac{(-1)^i}{i!\,(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^i = \frac{(1-1)^n}{n!} = \frac{0^n}{n!} = \boxed{0},$$ by the binomial theorem with $x = -1$, and because $0^n = 0$ for $n \geq 1$.	*This is where "positive integer n" is used: for $n = 0$ the sum equals 1.*
Check ($n = 2$). (a) $\frac{1}{0!\,2!} + \frac{1}{1!\,1!} + \frac{1}{2!\,0!} = \frac{1}{2} + 1 + \frac{1}{2} = 2 = \frac{2^2}{2!}$. (b) $\frac{1}{2} - 1 + \frac{1}{2} = 0$. ✓	

A combinatorial reading of (a)

Multiplying (a) by $n!$ gives $\sum_i \binom{n}{i} = 2^n$: the subsets of an n-set, sorted by their size. This identity was proved by a combinatorial argument in Tutorial 1; here it is used as a tool.

2.8 Problem 7: an inequality with five variables

Problem 7 (Exercises 1.4, #12)

Determine the number of integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 < 40$, where (A) $x_i \geq 0$, $1 \leq i \leq 5$; (B) $x_i \geq -3$, $1 \leq i \leq 5$.

How to think about it. Section 1.7. Turn the strict inequality into "≤ 39", add a slack variable, and use stars and bars with 6 variables. For (B), first shift the variables so that they start at 0.

Model solution to Problem 7(A)

Solution (what you write)	Commentary (what it does)
Statement. Count the 5-tuples of non-negative integers $(x_1, \ldots, x_5)$ with $x_1 + \cdots + x_5 < 40$.	
Solution. Since the x_i are integers, the condition is equivalent to $x_1 + \cdots + x_5 \leq 39$. Let $x_6 = 39 - (x_1 + \cdots + x_5)$. Then $x_6 \geq 0$ and $x_1 + \cdots + x_5 + x_6 = 39$, and conversely every non-negative solution of this equation gives a solution of the inequality (drop x_6). So we count non-negative solutions of an equation in $n = 6$ variables with right-hand side $r = 39$: by stars and bars, $\binom{6+39-1}{39} = \binom{44}{39} = \binom{44}{5}$.	*The "conversely" sentence justifies that the two counts are equal. x_6 is the slack.*
Answer. $\boxed{\binom{44}{5} = 1\,086\,008}$.	
Same count, other slack. With $x_6 = 40 - \sum x_i \geq 1$ and $y_6 = x_6 - 1 \geq 0$ one gets $x_1 + \cdots + x_5 + y_6 = 39$, the same equation. ✓	*This is the version in most student solutions.*

Model solution to Problem 7(B)

Solution (what you write)	Commentary (what it does)
Statement. Count the 5-tuples of integers with $x_i \geq -3$ for all i and $x_1 + \cdots + x_5 < 40$.	
Solution. Put $y_i = x_i + 3$ for $1 \leq i \leq 5$. Then $y_i \geq 0$, and $y_1 + \cdots + y_5 = x_1 + \cdots + x_5 + 15$, so the condition becomes $y_1 + \cdots + y_5 < 55$, that is, $y_1 + \cdots + y_5 \leq 54$. This substitution is a bijection between the two sets of solutions. As in (A), add the slack $y_6 = 54 - (y_1 + \cdots + y_5) \geq 0$ and count the non-negative solutions of $y_1 + \cdots + y_6 = 54$: $\binom{6+54-1}{54} = \binom{59}{5}$.	*The shift adds 3 to each of the 5 variables, so the bound goes up by 15.*
Answer. $\boxed{\binom{59}{5} = 5\,006\,386}$.	
Sanity check. Allowing negative values gives more solutions, and indeed $5\,006\,386 > 1\,086\,008$. ✓	*A quick comparison like this catches a shift in the wrong direction.*

3 Mistakes, checklist and exercises

3.1 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **Forgetting the order on the shelves.** In Problem 1, $2^{15} - 2$ only decides which books go on which shelf. The books on each shelf can still be permuted.
2. **Treating a round table like a row.** Seating the first group in $n!$ ways counts rotations as

different; using $m+1$ gaps instead of m lets two men meet in the "end" gap.

3. **Ordered versus unordered ranks.** Two pairs with $13 \cdot 12$ counts every hand twice; a full house with $\binom{13}{2}$ counts only half of the hands. Ask: does swapping the two ranks give a different hand?
4. **Hands that secretly upgrade.** For "three of a kind", choosing the last two cards freely from the 48 or 49 remaining cards lets in full houses or four of a kind. Name the exact pattern and make the extra cards obey it.
5. **Losing the constant term or the numbers in a multinomial.** Exponents add up to the power of the bracket; the constant is a block with its own exponent; every number is raised to its block's exponent, sign included.
6. **Using a binomial identity without naming it.** "$= 2^n/n!$" with no reason gets partial marks. Write "by the binomial theorem with $x = 1$".
7. **Strict inequality with a slack ≥ 0.** $x_1 + \cdots + x_5 + x_6 = 40$ with $x_6 \geq 0$ also counts the solutions with sum exactly 40. Use "≤ 39" or a slack ≥ 1.
8. **Shifting a negative bound the wrong way.** For $x_i \geq -3$, put $y_i = x_i + 3$; the right-hand side *increases* by 15.

Spot the error 1: Problem 1

"Solution." Each of the 15 books is put on the upper or the lower shelf: 2^{15} ways. Two of these leave a shelf empty (all books on one shelf). Answer: $2^{15} - 2 = 32\,766$.
Where is the mistake? (Answer in Appendix A.)

Spot the error 2: Problem 2

"Solution." Seat the women around the table: $8!$ ways. Write them as $\_W\_W\_W\_\cdots\_W\_$: there are 10 gaps. Choose 6 gaps for the men and arrange the men: $\binom{10}{6}6!$. Answer: $8!\binom{10}{6}6! = 6\,096\,384\,000$.
Where is the mistake? (Answer in Appendix A.) Hint: which two of the 10 gaps are next to each other at the table?

Spot the error 3: Problem 3(h)

"Solution." Choose the rank of the first pair (13) and its suits ($\binom{4}{2}$), then the rank of the second pair (12) and its suits ($\binom{4}{2}$), and finally the fifth card (44). Answer: $13 \cdot 6 \cdot 12 \cdot 6 \cdot 44 = 247\,104$.
Where is the mistake? (Answer in Appendix A.) Hint: compare with the correct answer 123 552.

Spot the error 4: Problem 5(b)

"Solution." The monomial $w^2x^2y^2z^2$ has degree 8, so its coefficient is $\binom{8}{2,2,2,2} \cdot 2^2(-1)^2 3^2 1^2 = 2520 \cdot 36 = 90\,720$.
Where is the mistake? (Answer in Appendix A.)

Spot the error 5: Problem 7(B)

"Solution." Put $y_i = x_i - 3$, so that $y_i \geq 0$. Then $\sum y_i = \sum x_i - 15 < 25$, i.e. $\sum y_i \leq 24$. With a slack: $\binom{6+24-1}{24} = \binom{29}{5} = 118\,755$.
Where is the mistake? (Answer in Appendix A.) Hint: compare with the answer to part (A).

3.2 Checklist before you hand in

Checklist

- ☐ I stated what is counted and what counts as "the same": order on a shelf? rotations at a table? a hand as a set?
- ☐ For shelves, I used **row + cuts** and checked whether empty shelves are allowed.
- ☐ For a round table, I seated one group in $(m-1)!$ ways and used m **gaps**, not $m+1$.
- ☐ For card hands, I wrote the **rank pattern**, chose ranks **in order** for different roles and **as a set** for equal roles, and made sure the hand cannot upgrade.
- ☐ For "at least" or "weight" conditions, I listed the **exact compositions** and explained why the list is complete.
- ☐ For a coefficient, my exponents add up to the **power of the bracket**, and I included the constant and every sign.
- ☐ For a sum, I named the **binomial theorem** and the value of x, and checked $n=2$.
- ☐ For an inequality, I converted "$<$" correctly, **shifted** the lower bounds, and added a **slack variable**.
- ☐ Every answer is **boxed** (expression and number).

3.3 Exercises

Write each solution in full, using the template.

E1. Pamela now has three shelves. In how many ways can she place her 15 books (a) so that every shelf has at least one book; (b) if shelves may be empty?

E2. (a) Six men and six women sit at a round table so that men and women alternate. How many seatings are there? (b) Solve Problem 2 again, but with the 15 people on a straight bench of 15 seats.

E3. (a) How many five-card hands have five different ranks? (b) How many five-card hands contain at least one ace? (c) Use (a), the one-pair example of Section 1.3 and Problem 3 to check that the six rank patterns add up to $\binom{52}{5}$.

E4. For the ternary strings of length 10: (a) how many have weight 5? (b) how many have at most two 0's?

E5. (a) Find the coefficient of x^3y^2z in $(x+2y-z+1)^8$. (b) What is the sum of *all* the coefficients in the expansion of $(2w-x+3y+z-2)^{12}$? (Hint: which values of w, x, y, z turn the expansion into the sum of its coefficients?)

E6. Evaluate, for every positive integer n: (a) $\displaystyle\sum_{i=0}^{n}\binom{n}{i}2^i$; (b) $\displaystyle\sum_{i=0}^{n}\frac{(-1)^i\,2^{n-i}}{i!\,(n-i)!}$.

E7. How many integer solutions does $x_1+x_2+x_3+x_4+x_5 \le 40$ have with $x_1 \ge 5$ and $x_i \ge 0$ for $2 \le i \le 5$?

E8. How many integer solutions does $x_1+x_2+x_3+x_4 < 20$ have with $x_i \ge -2$ for all i?

E9. Explain the answer $14 \cdot 15!$ of Problem 1 as $16!$ minus something, using a divider (Section 2.2), and generalise: show that n distinct books can be placed on two shelves with no shelf empty in $(n-1)\,n!$ ways.

E10. How many solutions of Problem 7 have all x_i *positive*?

A Answers to the "spot the error" puzzles

Spot the error 1 (no order on the shelves). The count $2^{15} - 2$ is the number of ways to decide *which* books go on the upper shelf (a non-empty proper subset). It ignores the order of the books on each shelf, which the question says matters. Once shelf 1 holds j books, they can be arranged in $j!$ ways and the others in $(15 - j)!$ ways. Adding these factors gives $\sum_{j=1}^{14} \binom{15}{j} j!\,(15-j)! = 14 \cdot 15!$.

Spot the error 2 (row gaps on a circle). Around a table the 9 women create only 9 gaps. The picture $\_\,W \cdots W\,\_$ is a row: its first and last gaps are the same place at the table (between the last and the first woman). Choosing both of them puts two men next to each other, so the "solution" includes bad seatings. Correct: $8!\binom{9}{6}6! = 2\,438\,553\,600$.

Spot the error 3 (ordered pair ranks). "First pair" and "second pair" are not different roles. The hand with pairs of 7's and K's is produced once with 7 first and once with K first, so every hand is counted exactly twice: $247\,104 = 2 \cdot 123\,552$. Choose the two pair ranks as a set, $\binom{13}{2}$, or divide by $2!$.

Spot the error 4 (missing constant). The bracket is raised to the power 12, so 12 blocks are chosen, not 8. The four remaining factors are the constant -2. The multinomial coefficient must be $\binom{12}{2,2,2,2,4}$ and the numbers must include $(-2)^4$: $1\,247\,400 \cdot 576 = 718\,502\,400$.

Spot the error 5 (shift in the wrong direction). From $x_i \geq -3$ we get $x_i + 3 \geq 0$, so the correct substitution is $y_i = x_i + 3$, and $\sum y_i = \sum x_i + 15 < 55$. The "solution" used $y_i = x_i - 3$, which describes $x_i \geq 3$. The sanity check catches it: $118\,755$ is *smaller* than the answer $1\,086\,008$ to part (A), although part (B) allows more tuples. Correct: $\binom{59}{5} = 5\,006\,386$.