Tutorial 2: Counting with Cases, Repetition, and Induction

Routes on a map, choosing with "at least", selections with repetition, and how to write an induction proof

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Count routes on a road map by splitting into disjoint route types (rule of sum) and multiplying along each type (rule of product), and count round trips with and without a "different return" condition.
2. Decide whether the *containers* (pots, rooms, teams) are identical or labelled, and choose between $\binom{n}{r}$ and $P(n,r)$ accordingly.
3. Handle "at least k of each kind" by splitting into cases according to the exact composition, and recognise the classic overcount that comes from "choose the minimum first, then the rest".
4. Count selections with repetition and integer solutions of $x_1 + \cdots + x_n = r$ with the stars-and-bars formula $\binom{n+r-1}{r}$, including lower bounds on the variables.
5. Write a proof by mathematical induction in the standard format: **Statement** → **Base case** → **Inductive hypothesis** → **Inductive step** → **Conclusion**, for sums, divisibility statements and inequalities.

Sections 1.1 to 1.6 review the tools with a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 solves all seven tutorial problems in full. Section 3 collects the classic mistakes, a checklist and extra exercises. Tutorial 1 (sum and product rules, $P(n,r)$, $\binom{n}{r}$) is assumed.

Contents

1 The toolbox

The first three problems of this tutorial are counting problems that combine the Tutorial 1 tools in new ways; the last four are proofs by induction. As before, every tool is explained the same way: the idea in plain words, a small example, the precise statement, a bigger example, and a recap.

1.1 Sum and product on a road map

The intuition

A trip on a map is a sequence of roads. To count trips, **sort them by the towns they pass through**. Inside one sort, the trip is built leg by leg, so you **multiply** the number of roads on each leg ("this road *and then* that road"). Different sorts never produce the same trip, so you **add** over the sorts ("either through B *or* directly").

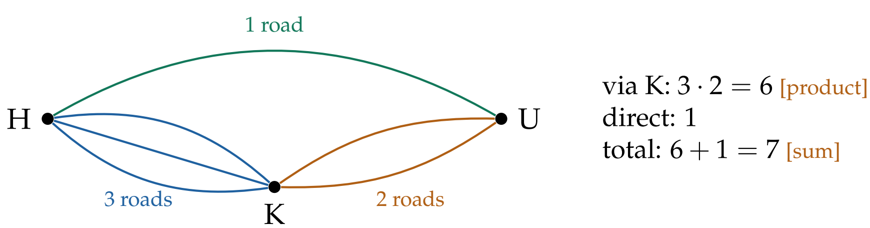


Figure 1. Home (H), a café (K) and the university (U). Sort the trips from H to U by whether they pass through K.

Simple example: home to campus

In Figure 1 there are 3 roads from home H to the café K, 2 roads from K to the university U, and 1 road from H straight to U. A trip from H to U either passes through K or it does not.

- Through K: choose a road H–K (3 ways) *and then* a road K–U (2 ways): $3 \cdot 2 = 6$ trips [rule of product].
- Directly: 1 trip.

The two sorts do not overlap, so there are $6 + 1 = 7$ trips [rule of sum].

Counting trips and round trips

Let N be the number of trips from X to Y (we assume a trip never visits a town twice).

(i) **One-way trips.** Split the trips into disjoint *route types* (the list of towns visited, in order). For each type, multiply the numbers of roads on its legs; then add over the types.

(ii) **Round trips.** A round trip $X \to Y \to X$ is an outbound trip *and then* a return trip. With two-way roads the return trips are the N trips read backwards, so there are $N \cdot N = N^2$ round trips.

(iii) **Round trips with a different return.** If the return may be anything *except* the outbound trip reversed, then for each of the N outbound trips there are $N-1$ returns, giving $N(N-1)$. Equivalently, by the complement: N^2 minus the N round trips whose return repeats the outbound route.

Comprehensive example: four towns

Towns P, Q, R, S are joined by two-way roads: 3 roads P–Q, 2 roads Q–S, 2 roads P–R, 4 roads R–S, and 1 road P–S. There are no roads Q–R. (a) How many trips from P to S? (b) How many round trips P $\to$ S $\to$ P? (c) How many round trips in which the return is not the outbound trip reversed?

Solution. (a) The route types are P–Q–S, P–R–S and P–S; a trip has exactly one type. By the rule of product the types contain $3 \cdot 2 = 6$, $2 \cdot 4 = 8$ and 1 trips, and by the rule of sum $N = 6 + 8 + 1 = 15$. (b) Outbound and then return: $15 \cdot 15 = 225$. (c) For each outbound trip, 14 returns remain: $15 \cdot 14 = 210$. Check: $225 - 15 = 210$. ✓

Add the route types, do not multiply them

A tempting wrong answer to (a) is $3 \cdot 2 \cdot 2 \cdot 4 \cdot 1$: it multiplies numbers that belong to *different* trips. A single trip uses the P–Q roads *or* the P–R roads, never both, so the route types are added.

Maps in one breath

- Sort trips by route type; multiply along a type, add over the types.
- Round trips: N^2. Round trips whose return differs from the outbound route: $N(N-1) = N^2 - N$.

1.2 Identical or labelled containers?

The intuition

When distinct objects are placed into containers, ask: **can I tell the containers apart?** If the containers are labelled (Pot 1, Pot 2, Pot 3), then putting the tulip in Pot 1 and the lily in Pot 2 is different from the reverse, so the *order* of the chosen objects matters. If the containers are identical, the only thing you can see is *which* objects were used, so order does not matter.

Simple example: two books on two shelves

Pick 2 of 5 different books and put one on each of two shelves.

- Labelled shelves (top and bottom): $5 \cdot 4 = P(5,2) = 20$ ways.
- Identical shelves (you only see "these two books are on display"): $\binom{5}{2} = 10$ ways.

Each unordered pair appears twice in the labelled count, once for each way of assigning the two books to the shelves: $20 = 10 \cdot 2!$.

One object per container

Let $1 \leq r \leq n$. Choose r of n distinct objects and place them into r containers, one object per container.

- If the containers are **labelled**, the number of ways is $P(n,r) = \dfrac{n!}{(n-r)!}$.
- If the containers are **identical**, the number of ways is $\binom{n}{r} = \dfrac{P(n,r)}{r!}$.

The factor $r!$ is the number of ways to permute the chosen objects among the containers; when the containers are identical, all $r!$ of those permutations look the same.

Comprehensive example: TAs in pairs, labelled or not

Six TAs are split into three pairs. (a) The pairs are assigned to the labelled time slots A, B, C. (This was the Tutorial 1 example.) (b) The pairs are just study groups, with no labels.
Solution. (a) Choose the pair for A, then for B, then for C: $\binom{6}{2}\binom{4}{2}\binom{2}{2} = 15 \cdot 6 \cdot 1 = 90$.
(b) Each unlabelled split into pairs, such as $\{12\}, \{34\}, \{56\}$, appears in (a) exactly $3! = 6$ times, once for each way to hand the three pairs to the three slots. So the number of unlabelled splits is $90/3! = 15$.
Check. Pair TA 1 with one of the other 5; then the smallest remaining TA with one of the other 3; the last two form the third pair: $5 \cdot 3 \cdot 1 = 15$. ✓

Containers in one breath

- Labelled containers: order matters, $P(n,r)$. Identical containers: only the set matters, $\binom{n}{r}$.
- Going from labelled to identical, divide by the number of ways to permute the containers ($r!$).

1.3 "At least" conditions: split by the exact composition

The intuition

"At least 2 of each kind" allows many different mixes. Instead of fighting the words "at least", **list the possible exact mixes**. Each mix is an easy product of binomial coefficients, and the mixes do not overlap, so you add them.

Simple example: a committee

A committee of 5 is chosen from 4 CS students and 4 Math students, with at least 2 from each program. The possible exact mixes (CS, Math) are $(3,2)$ and $(2,3)$; $(4,1)$ and $(1,4)$ break the condition. Hence

$$\binom{4}{3}\binom{4}{2} + \binom{4}{2}\binom{4}{3} \;=\; 4\cdot 6 + 6\cdot 4 \;=\; 48.$$

Splitting by composition

Suppose objects come in kinds, kind i containing n_i distinct objects, and we choose r objects in total subject to bounds on how many come from each kind. Then

$$\#\text{selections} \;=\; \sum_{(k_1,\ldots,k_t)} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_t}{k_t},$$

where the sum runs over all *compositions* $(k_1,\ldots,k_t)$ with $k_1+\cdots+k_t = r$ that satisfy the bounds (and $0 \le k_i \le n_i$). Different compositions give different selections, which is why the rule of sum applies.

The "minimum first" overcount

A very common wrong method for the committee: "Choose 2 CS students ($\binom{4}{2}$), choose 2 Math students ($\binom{4}{2}$), then choose the fifth member from the 4 people left (4)", giving $6\cdot 6\cdot 4 = 144$. This is 3 times too big. Take the committee $\{c_1,c_2,c_3,m_1,m_2\}$: it is produced when the "guaranteed" CS pair is $\{c_1,c_2\}$ and the extra is c_3, but also for $\{c_1,c_3\}$ with extra c_2 and $\{c_2,c_3\}$ with extra c_1. The procedure does not produce each committee exactly once, so the rule of product does not count committees.

Comprehensive example: a hiring panel

A panel of 6 is formed from 5 professors, 4 graduate students and 3 undergraduates (all different people). The panel must contain at least one person from each group and at most 2 professors. How many panels are there?

Solution. Write (p,g,u) for the exact numbers of professors, graduate students and undergraduates, so $p+g+u = 6$, $1 \le p \le 2$, $1 \le g \le 4$, $1 \le u \le 3$.

If $p = 1$ then $g+u = 5$, so $(g,u) \in \{(2,3),(3,2),(4,1)\}$: $\binom{5}{1}\left[\binom{4}{2}\binom{3}{3} + \binom{4}{3}\binom{3}{2} + \binom{4}{4}\binom{3}{1}\right] = 5\,(6+12+3) = 105$.

If $p = 2$ then $g+u = 4$, so $(g,u) \in \{(1,3),(2,2),(3,1)\}$: $\binom{5}{2}\left[\binom{4}{1}\binom{3}{3} + \binom{4}{2}\binom{3}{2} + \binom{4}{3}\binom{3}{1}\right] = 10\,(4+18+12) = 340$.

The cases are disjoint, so the answer is $105 + 340 = \boxed{445}$.

"At least" in one breath

- List the exact compositions allowed by the bounds, count each as a product of binomials, add.
- Never "choose the minimum first and then the rest" for distinct objects: it counts some selections several times.

1.4 Selections with repetition: stars and bars

The intuition

Put 3 identical bananas into 10 numbered boxes, several bananas allowed in a box. An outcome is determined by *how many* bananas each box gets. Line the boxes up and draw a bar between consecutive boxes: 10 boxes need 9 bars. Now write a star $\star$ for each banana, inside its box. Every outcome becomes a row of 3 stars and 9 bars, and every such row is an outcome. So we only need to **choose which** 3 **of the** 12 **positions hold the stars**.

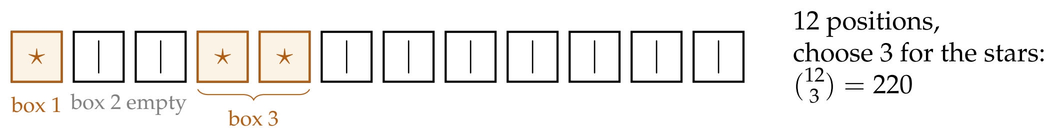


Figure 2. One banana in box 1*, two in box* 3*, none elsewhere. The* 9 *bars separate the* 10 *boxes, and the stars between the* $(j-1)$*st and* j*th bar are the bananas in box* j*.*

Simple example: bananas in boxes (from the lecture)

In the lecture this was first counted by cases: all three bananas in different boxes $\binom{10}{3}$, two together and one apart $10 \cdot 9$, all three together 10. Total $120 + 90 + 10 = 220$. Stars and bars gives the same number at once: $\binom{3+10-1}{3} = \binom{12}{3} = 220$. ✓

Selections with repetition and integer solutions

Let $n \geq 1$ and $r \geq 0$ be integers.

(i) The number of ways to select r objects from n distinct kinds, *repetition allowed and order irrelevant* (equivalently, to distribute r identical objects into n distinct boxes), is

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

(ii) Equivalently, the number of integer solutions of $x_1 + x_2 + \cdots + x_n = r$ with every $x_i \geq 0$ is $\binom{n+r-1}{r}$. (Here x_i is the number of objects of kind i; r stars, $n-1$ bars.)

(iii) **Positive solutions.** The number of solutions with every $x_i \geq 1$ is $\binom{r-1}{n-1}$ (for $r \geq n$). Reason 1: give each x_i its guaranteed 1 first, then distribute the remaining $r-n$ freely: $\binom{n+(r-n)-1}{r-n} = \binom{r-1}{r-n} = \binom{r-1}{n-1}$. Reason 2: line up r stars; there are $r-1$ gaps between them; choose $n-1$ of the gaps for the bars (at most one bar per gap, so no part is empty).

(iv) **Lower bounds in general.** If $x_i \geq b_i$, substitute $y_i = x_i - b_i \geq 0$. The equation becomes $y_1 + \cdots + y_n = r - (b_1 + \cdots + b_n)$, and (ii) applies.

Select r from n distinct kinds	**order matters**	**order does not matter**
no repetition	$P(n,r) = \dfrac{n!}{(n-r)!}$	$\dbinom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$
repetition allowed	n^r	$\dbinom{n+r-1}{r}$

Table 1. The four basic ways to select r things from n. Tutorial 1 covered three of the cells; stars and bars completes the table.

Comprehensive example: a dozen donuts

A shop sells 5 kinds of donuts (plenty of each). How many ways are there to buy 12 donuts (a) with no restriction; (b) with at least one of each kind; (c) with at least 3 chocolate donuts? Donuts of the same kind are identical, and the order of purchase does not matter.
Solution. Let x_i be the number of donuts of kind i, so we count solutions of $x_1 + \cdots + x_5 = 12$.
(a) $x_i \geq 0$: $\binom{5+12-1}{12} = \binom{16}{12} = \binom{16}{4} = 1820$.
(b) $x_i \geq 1$: put one of each kind in the bag first; 7 donuts remain to be chosen freely: $\binom{5+7-1}{7} = \binom{11}{4} = 330$.
(c) $x_1 \geq 3$ (kind 1 is chocolate): substitute $y_1 = x_1 - 3$, giving $y_1 + x_2 + \cdots + x_5 = 9$: $\binom{5+9-1}{9} = \binom{13}{4} = 715$.

Which number is n and which is r?

In $\binom{n+r-1}{r}$, n is the number of *kinds* (boxes, variables) and r is the number of *objects* (stars, the right-hand side). A quick self-test: the bottom of the equivalent form $\binom{n+r-1}{n-1}$ is the number of *bars*, one fewer than the number of boxes.

Stars and bars in one breath

- r identical objects into n boxes = solutions of $x_1 + \cdots + x_n = r$, $x_i \geq 0 = \binom{n+r-1}{r}$.
- Positive solutions: $\binom{r-1}{n-1}$. Other lower bounds: subtract them first, then use the formula.

1.5 Mathematical induction

The intuition

Picture an infinite row of dominoes numbered $1, 2, 3, \ldots$. You are sure they all fall if you know two things: **the first domino falls**, and **whenever a domino falls, it knocks over the next one**. You never check domino 1000 directly; the chain $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow \cdots \Rightarrow 1000$ does it for you. Induction proves a statement $P(n)$ for *every* n in exactly this way.

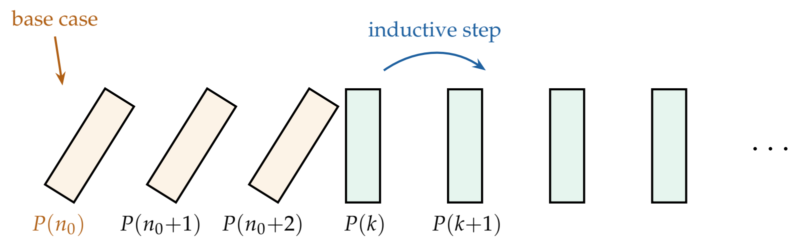


Figure 3. The base case knocks over the first domino; the inductive step says that each domino knocks over the next.

Simple example: $1+2+\cdots+n=\frac{n(n+1)}{2}$

Let $P(n)$ be the statement $1+2+\cdots+n=\dfrac{n(n+1)}{2}$.

Base case. For $n=1$: the left side is 1 and the right side is $\frac{1\cdot 2}{2}=1$. So $P(1)$ holds.

Inductive hypothesis. Let $k\geq 1$ and assume $P(k)$: $1+2+\cdots+k=\dfrac{k(k+1)}{2}$.

Inductive step. We show $P(k+1)$: $1+2+\cdots+(k+1)=\dfrac{(k+1)(k+2)}{2}$. Indeed,

$$1+\cdots+k+(k+1)\overset{\text{IH}}{=}\frac{k(k+1)}{2}+(k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

Conclusion. By the principle of mathematical induction, $P(n)$ holds for every integer $n\geq 1$. ■

The principle of mathematical induction

Let $P(n)$ be a statement about the integer n, and let n_0 be a fixed integer. Suppose

(i) **(Base case)** $P(n_0)$ is true, and

(ii) **(Inductive step)** for every integer $k\geq n_0$: *if* $P(k)$ is true, *then* $P(k+1)$ is true.

Then $P(n)$ is true for every integer $n\geq n_0$.

The fine print.

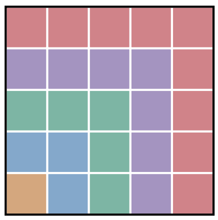
- The assumption "$P(k)$ is true" is called the **inductive hypothesis** (IH). It is an assumption about *one arbitrary* $k\geq n_0$, not about all n; assuming it for all n would be assuming what we want to prove.
- The base case n_0 need not be 1. It is the first value the statement is claimed for: $n_0=0$ for "every $n\geq 0$", $n_0=4$ for "every $n\geq 4$".
- Both parts are needed. A correct inductive step with no base case proves nothing (Section 3, spot the error 2).

Comprehensive example: the sum of the first n odd numbers

Statement. For every integer $n \geq 1$, $1+3+5+\cdots+(2n-1) = n^2$.
Base case. $n=1$: the left side has one term, 1, and $1^2 = 1$. ✓
Inductive hypothesis. Let $k \geq 1$ and assume $1+3+\cdots+(2k-1) = k^2$.
Inductive step. The goal is $1+3+\cdots+(2k-1)+(2k+1) = (k+1)^2$. The last term for $n=k+1$ is $2(k+1)-1 = 2k+1$, so

$$\underbrace{1+3+\cdots+(2k-1)}_{=k^2 \text{ by IH}} + (2k+1) = k^2+2k+1 = (k+1)^2.$$

Conclusion. By induction, the identity holds for all $n \geq 1$. ■



$1+3+5+7+9 = 5^2$
the kth L-shaped layer has $2k-1$ squares;
adding the layer $2k+1$ turns the $k \times k$ square into the $(k+1)\times(k+1)$ square.

Figure 4. The inductive step, as a picture: one more L-shaped layer of $2k+1$ squares.

Induction in one breath

- Prove $P(n_0)$; then, for an arbitrary $k \geq n_0$, assume $P(k)$ and derive $P(k+1)$; conclude $P(n)$ for all $n \geq n_0$.
- Write the goal $P(k+1)$ down explicitly before you start the step, and mark the place where you use the IH.

1.6 The three shapes of induction proofs

The intuition

The base case is always a direct check. The inductive step is where the work is, and in COMP 339 it almost always has one of three shapes. In each, the trick is the same: **rewrite the $(k+1)$ case so that the k case appears inside it**, then replace it using the IH.

Shape	**How to find $P(k)$ inside $P(k+1)$**	**Typical line**
Sum = formula	split off the last term	(sum up to k) + last term $\stackrel{\text{IH}}{=}$ formula(k) + last term
$d \mid f(n)$	write the IH as $f(k) = d \cdot a$; express $f(k+1)$ as (multiple of $f(k)$) + (multiple of d)	$5^{k+1}-1 = 5(5^k-1)+4 = 5\cdot 4a+4$
$f(n) > g(n)$	start from $f(k+1)$, use the IH once, then use $k \geq n_0$ to finish	$(k+1)! = (k+1)\,k! > (k+1)\,2^k \geq \cdots$

Table 2. The three shapes. Problem 7 has the first shape, Problems 4 and 6 the second, and Problem 5 the third.

Simple example (divisibility): $6 \mid 7^n - 1$ for all $n \geq 0$

Base case. $n = 0$: $7^0 - 1 = 0 = 6 \cdot 0$. ✓
Inductive hypothesis. Let $k \geq 0$ and assume $6 \mid 7^k - 1$, that is, $7^k - 1 = 6a$ for some integer a.
Inductive step. Add and subtract 7 to make $7^k - 1$ appear:

$$7^{k+1} - 1 \;=\; 7 \cdot 7^k - 7 + 7 - 1 \;=\; 7\,(7^k - 1) + 6 \;\overset{\text{IH}}{=}\; 7 \cdot 6a + 6 \;=\; 6\,(7a + 1).$$

Since $7a + 1$ is an integer, $6 \mid 7^{k+1} - 1$. **Conclusion.** By induction, $6 \mid 7^n - 1$ for all $n \geq 0$. ∎

Divisibility, precisely

For integers $d \neq 0$ and m, "d divides m", written $d \mid m$ (the lecture also writes $m \,\vdots\, d$), means that $m = d \cdot a$ for some *integer* a. Two facts do all the work in divisibility inductions: if $d \mid m_1$ and $d \mid m_2$, then $d \mid m_1 + m_2$; and if $d \mid m$, then $d \mid c \cdot m$ for every integer c.

Comprehensive example (inequality): $2^n > n^2$ for all $n \geq 5$

Why $n \geq 5$? $2^4 = 16 = 4^2$ and $2^3 = 8 < 9$, so the claim fails for $n = 3, 4$; the base case must be $n_0 = 5$.
Base case. $n = 5$: $2^5 = 32 > 25 = 5^2$. ✓
Inductive hypothesis. Let $k \geq 5$ and assume $2^k > k^2$.
Inductive step. Goal: $2^{k+1} > (k+1)^2$. We have

$$2^{k+1} \;=\; 2 \cdot 2^k \;\overset{\text{IH}}{>}\; 2k^2 \;=\; k^2 + k^2 \;\geq\; k^2 + 2k + 1 \;=\; (k+1)^2,$$

where the "$\geq$" uses $k^2 \geq 2k + 1$, which holds because $k^2 - 2k - 1 = (k-1)^2 - 2 \geq 16 - 2 > 0$ for $k \geq 5$.
Conclusion. By induction, $2^n > n^2$ for all $n \geq 5$. ∎

Inequalities: use the IH once, then the range of k

In an inequality step you usually need a small extra fact, like $k^2 \geq 2k + 1$ above or $k + 1 > 2$ in Problem 5. That extra fact is where the condition $k \geq n_0$ is used. If your step never uses $k \geq n_0$, check that the statement really needs that starting point, or look for the gap in your argument.

The three shapes in one breath

- Sums: split off the last term. Divisibility: write the IH as "$= d \cdot a$" and add and subtract to make $f(k)$ appear.
- Inequalities: start from the bigger side of $P(k+1)$, use the IH once, finish with $k \geq n_0$.

The toolbox at a glance

Situation	Count / method	Signal words
Trips on a map	multiply along a route type, add over types	"roads", "through B or directly"
Round trips; different return	N^2; $N(N-1)$	"and back", "at least partially different"
r distinct objects into r identical containers	$\binom{n}{r}$	"identical pots", "unlabelled"
r distinct objects into r labelled containers	$P(n, r)$	"numbered", "slots", "Pot 1"
"At least k of each kind" (distinct objects)	sum over exact compositions	"at least", "at most", "must use"
r identical objects into n boxes; $x_1 + \cdots + x_n = r$, $x_i \geq 0$	$\binom{n+r-1}{r}$	"identical", "repetition allowed", "integer solutions"
Same, with $x_i \geq 1$	$\binom{r-1}{n-1}$	"positive", "at least one in each"
Statement for all $n \geq n_0$	induction: base, IH, step	"prove by induction", "for all n"

2 The tutorial problems, worked in full

2.1 How to write the solutions

Problems 1 to 3 are counting problems, written with the Tutorial 1 template: **Statement** → **Solution** (set-up, then a step-by-step count naming the rule used at each step) → **Answer**, with an optional *Check*. Problems 4 to 7 are induction proofs, and they have their own template. Graders look for the same things every time: a clearly stated $P(n)$, a base case that is actually verified, an inductive hypothesis that is *assumed* (not proved), an inductive step whose goal is written out and where the use of the IH is visible, and a concluding sentence.

Writing template for a proof by induction

Statement. *[Define $P(n)$ and the range: "For every integer $n \geq n_0$, …". Say "We prove this by induction on n."]*
Base case. *[Check $P(n_0)$ by direct computation. Show both sides as numbers.]*
Inductive hypothesis. *["Let $k \geq n_0$ be an arbitrary integer and assume $P(k)$, that is, …" Write $P(k)$ out in full. For divisibility, write it as an equation: "$\cdots = d \cdot a$ for some integer a".]*
Inductive step. *["We show $P(k+1)$, that is, …" Write the goal out in full. Then start from one side of the goal and transform it until you reach the other side (or the required divisibility or inequality). Mark the step that uses the IH.]*
Conclusion. *["By the principle of mathematical induction, $P(n)$ holds for every integer $n \geq n_0$."]*

Phrasebook: sentences that belong in an induction proof

Where	What to write
Setting up	"Let $P(n)$ denote the statement ..." "We proceed by induction on $n \geq n_0$."
Base case	"For $n = n_0$ the left-hand side is ... and the right-hand side is ..., so $P(n_0)$ holds."
Hypothesis	"Let $k \geq n_0$ and suppose that ... (inductive hypothesis)." "..., i.e. $f(k) = d\,a$ for some $a \in \mathbb{Z}$."
Goal	"We must show that ..." "It remains to prove $P(k+1)$: ..."
Using the IH	"By the inductive hypothesis, ..." "$\overset{\text{IH}}{=}$" "Substituting $f(k) = d\,a$, ..."
Divisibility	"Both terms are multiples of d, hence so is their sum." "Since $5a+1$ is an integer, ..."
Inequality	"Since $k \geq 4$, we have $k+1 > 2$, and therefore ..."
Finishing	"This proves $P(k+1)$." "By the principle of mathematical induction, ... for all $n \geq n_0$."

2.2 Problem 1: Linda's road trips

Problem 1 (Problem 11, Ex. 1.1 and 1.2)

Three small towns, designated by A, B, and C, are interconnected by a system of two-way roads, as shown in Figure 5.

a) In how many ways can Linda travel from town A to town C?
b) How many different round trips can Linda travel from town A to town C and back to town A?
c) How many of the round trips in part (b) are such that the return trip (from town C to town A) is at least partially different from the route Linda takes from town A to town C?

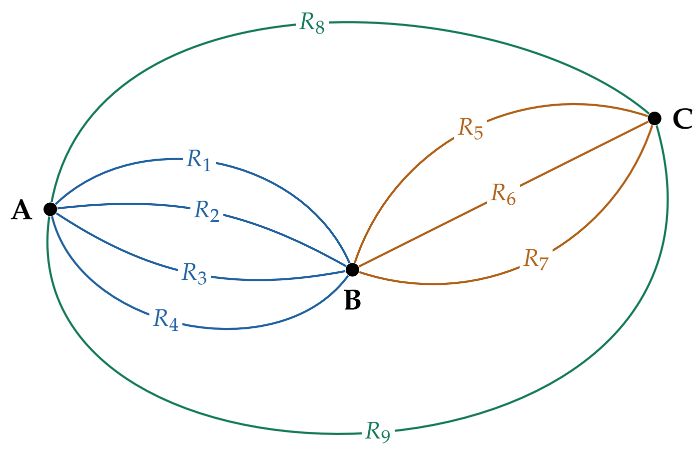


Figure 5. The road system: $R_1, \ldots, R_4$ join A and B, R_5, R_6, R_7 join B and C, and R_8, R_9 join A and C directly.

How to think about it. Read the map as three bundles of roads: 4 between A and B, 3 between B and C, 2 between A and C. A trip from A to C either goes through B or goes directly, and these are the only two route types (a trip does not revisit a town). This is exactly Section 1.1.

Model solution to Problem 1(a)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the trips from A to C along the roads of Figure 5, where a trip visits each town at most once.	*State the assumption: without it, Linda could drive back and forth forever and the count would be infinite.*
Solution. Every trip from A to C is of exactly one of two types. *Type 1: through B.* Linda chooses one of the 4 roads from A to B, and then one of the 3 roads from B to C. By the rule of product there are $4 \cdot 3 = 12$ such trips. *Type 2: directly.* Linda takes R_8 or R_9: 2 trips. The types are disjoint (a trip either passes through B or it does not), so by the rule of sum the number of trips is $12 + 2$.	*Sort by route type, multiply along each type, add over the types. Saying why the types are disjoint justifies the rule of sum.*
Answer. $\boxed{2 + 4 \cdot 3 = 14}$.	
Check. List by the first road: from $R_1, \ldots, R_4$ there are 3 ways to finish each (12), from R_8 or R_9 the trip is already over (2). ✓	*Counting in a different order gives the same number.*

Model solution to Problem 1(b)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the round trips $A \to C \to A$, a round trip being an outbound trip from A to C followed by a return trip from C to A.	
Solution. By part (a) there are 14 choices for the outbound trip. The roads are two-way, so the trips from C to A are the 14 trips of part (a) traversed backwards, and any of them may be used for the return, whatever the outbound trip was. By the rule of product the number of round trips is $14 \cdot 14$.	*The number of return options (14) does not depend on the outbound choice, so the rule of product applies.*
Answer. $\boxed{(2 + 4 \cdot 3)^2 = 14^2 = 196}$.	

Model solution to Problem 1(c)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the round trips from part (b) whose return trip is not simply the outbound trip traversed backwards.	*"At least partially different" means "not identical". The only forbidden return is the exact reverse of the outbound trip.*
Solution. Choose the outbound trip: 14 ways. For each outbound trip, exactly one of the 14 return trips is the same route reversed, so $14 - 1 = 13$ return trips are allowed. By the rule of product the count is $14 \cdot 13$.	*Same count by the complement: all round trips minus the 14 that repeat the route: $196 - 14$.*
Answer. $\boxed{14 \cdot 13 = 182}$.	

Check. $196 - 14 = 182$. ✓

Partially different vs completely different

If instead the return trip may use *no road at all* from the outbound trip, the count changes, because the number of allowed returns now depends on the outbound trip. You must split into cases: a direct outbound trip leaves $1 + 12 = 13$ returns, and an outbound trip through B leaves $2 + 3 \cdot 2 = 8$ returns. Hence $2 \cdot 13 + 12 \cdot 8 = 122$ round trips. (Exercise E1.)

2.3 Problem 2: the florist

Problem 2

a) A florist has 7 different roses and 3 identical flowerpots. Count the number of ways she can plant a rose in each of the flowerpots.
b) A florist has 6 Roses, 5 Tulips and 4 Lotuses. All the flowers of each kind are different. In how many ways can the florist make a bouquet of 10 flowers if she must use at least 3 of each kind?

How to think about it. Part (a) is Section 1.2: distinct roses, identical pots, one per pot. Part (b) is Section 1.3: distinct flowers, "at least 3 of each kind", so list the exact compositions. Since $3 + 3 + 3 = 9$, exactly one kind gets a fourth flower, and that gives three cases.

Model solution to Problem 2(a)

Solution (what you write)	Commentary (what it does)
Statement. Count the ways to choose 3 of 7 distinct roses and plant one in each of 3 identical pots.	*Two facts decide the tool: the roses are different, the pots are not.*
Solution. Since the pots are identical, an outcome is determined only by *which* three roses are planted; permuting the chosen roses among the pots gives the same outcome. So we count 3-element subsets of the 7 roses: $\binom{7}{3}$.	*Identical containers, one object each:* $\binom{n}{r}$.
Answer. $\boxed{\binom{7}{3} = 35}$.	
Check. With labelled pots the count would be $P(7, 3) = 210$, and each set of three roses would appear $3! = 6$ times: $210/6 = 35$. ✓	*Relate the two situations by the factor* $r!$.

Case	Roses	Tulips	Lotuses	Check	Number of bouquets
1	4	3	3	$4+3+3=10$	$\binom{6}{4}\binom{5}{3}\binom{4}{3} = 15\cdot 10\cdot 4 = 600$
2	3	4	3	$3+4+3=10$	$\binom{6}{3}\binom{5}{4}\binom{4}{3} = 20\cdot 5\cdot 4 = 400$
3	3	3	4	$3+3+4=10$	$\binom{6}{3}\binom{5}{3}\binom{4}{4} = 20\cdot 10\cdot 1 = 200$
					total $= 1200$

Table 3. Problem 2(b): the three exact compositions. Every allowed bouquet is in exactly one row.

Model solution to Problem 2(b)

Solution (what you write)	Commentary (what it does)
Statement. Count the 10-element sets of flowers chosen from 6 distinct roses, 5 distinct tulips and 4 distinct lotuses that contain at least 3 flowers of each kind.	*A bouquet is a set: the order of the flowers does not matter.*
Solution. Let (r,t,ℓ) be the numbers of roses, tulips and lotuses in the bouquet. Then $r+t+\ell = 10$ and $r,t,\ell \geq 3$. Since $3\cdot 3 = 9$, exactly one of r,t,ℓ equals 4 and the other two equal 3, giving the three disjoint cases $(4,3,3)$, $(3,4,3)$, $(3,3,4)$ (all allowed, since $4 \leq 6,5,4$). In each case we choose the roses, then the tulips, then the lotuses, so by the rule of product: $(4,3,3)$: $\binom{6}{4}\binom{5}{3}\binom{4}{3} = 600$; $(3,4,3)$: $\binom{6}{3}\binom{5}{4}\binom{4}{3} = 400$; $(3,3,4)$: $\binom{6}{3}\binom{5}{3}\binom{4}{4} = 200$. By the rule of sum the total is $600+400+200$.	*List the exact compositions first, and explain why the list is complete. Then one product of binomials per case.*
Answer. $\binom{6}{4}\binom{5}{3}\binom{4}{3} + \binom{6}{3}\binom{5}{4}\binom{4}{3} + \binom{6}{3}\binom{5}{3}\binom{4}{4}$ $\boxed{= 600+400+200 = 1200}$.	*Give the expression (graders can check it term by term) and the number.*

Do not do this

"Pick 3 roses, 3 tulips, 3 lotuses, then any 1 of the remaining 6 flowers" gives $\binom{6}{3}\binom{5}{3}\binom{4}{3}\cdot 6 = 4800$, four times the right answer. See spot the error 1 in Section 3 for why the factor is exactly 4.

2.4 Problem 3: integer solutions

Problem 3 (Problem 7, Ex. 1.4)

Determine the number of integer solutions of $x_1 + x_2 + x_3 + x_4 = 32$, where
a) $x_i > 0, 1 \le i \le 4$;
b) $x_i \ge 0, 1 \le i \le 4$.

How to think about it. A solution is a way to split 32 identical stars into 4 ordered parts, so this is stars and bars (Section 1.4) with $n = 4$ variables and $r = 32$. Part (b) is the basic formula; part (a) is the positive version.

$$\underbrace{\star\star\star\star\star}_{x_1=5}\,|\,\underbrace{\star\star\star\star\star\star\star\star\star}_{x_2=9}\,|\,\underbrace{\star\star\star\star\star\star\star}_{x_3=7}\,|\,\underbrace{\star\star\star\star\star\star\star\star\star\star\star}_{x_4=11}$$

Figure 6. The solution $(5, 9, 7, 11)$. For (a) the 3 bars go into 3 of the 31 gaps between *stars (no empty part); for (b) the 3 bars and 32 stars are arranged freely in 35 positions.*

Model solution to Problem 3(a)

Solution (what you write)	Commentary (what it does)
Statement. Count the 4-tuples (x_1, x_2, x_3, x_4) of positive integers with $x_1 + x_2 + x_3 + x_4 = 32$.	*Solutions are ordered: $(1, 1, 1, 29)$ and $(29, 1, 1, 1)$ are different.*
Solution. Substitute $y_i = x_i - 1$. Then $y_i \ge 0$ and $y_1 + y_2 + y_3 + y_4 = 32 - 4 = 28$, and this substitution is a one-to-one correspondence between the positive solutions of the original equation and the non-negative solutions of the new one. By stars and bars with $n = 4$ and $r = 28$, the number of non-negative solutions is $\binom{4+28-1}{28} = \binom{31}{28} = \binom{31}{3}$.	*Remove the lower bound first, and say why the new count equals the old one (the substitution is a bijection).*
Answer. $\boxed{\binom{31}{3} = 4495}$.	
Check. Directly: 32 stars in a row have 31 gaps between them; choosing 3 gaps for the bars splits the stars into 4 non-empty parts: $\binom{31}{3}$. ✓	*This is the formula $\binom{r-1}{n-1}$ from Section 1.4.*

Model solution to Problem 3(b)

Solution (what you write)	Commentary (what it does)
Statement. Count the 4-tuples of non-negative integers with $x_1 + x_2 + x_3 + x_4 = 32$.	
Solution. A solution corresponds to a row of 32 stars and 3 bars: the number of stars before the first bar is x_1, between the first and second bar x_2, and so on; conversely every such row gives a solution. A row is determined by the positions of the 3 bars among the 35 positions, so the number of solutions is $\binom{35}{3}$.	*Explain the correspondence once; after that, the formula $\binom{n+r-1}{r}$ may simply be quoted.*

Answer. $\boxed{\binom{35}{3} = 6545}$.

Check. Positive solutions are fewer: $4495 < 6545$. ✓ Also, the number of solutions of (b) with some $x_i = 0$ is $6545 - 4495 = 2050$.

A sanity check on the relative sizes.

2.5 Problem 4: $4 \mid 5^n - 1$

Problem 4

Prove by induction that for any $n \in \mathbb{N}$ the number $5^n - 1$ is divisible by 4. Explicitly write which is the base of induction and which is the inductive step.

How to think about it. Divisibility shape (Section 1.6). Write the IH as $5^k - 1 = 4a$ and make $5^k - 1$ appear inside $5^{k+1} - 1$ by adding and subtracting 5: $5^{k+1} - 1 = 5 \cdot 5^k - 5 + 4$.

Model solution to Problem 4

Solution (what you write)	Commentary (what it does)
Statement. For every integer $n \geq 1$, $4 \mid 5^n - 1$. We prove this by induction on n.	*Here $\mathbb{N} = \{1, 2, 3, \ldots\}$, so $n_0 = 1$. (If your convention is $0 \in \mathbb{N}$, see the remark below.)*
Base case. For $n = 1$: $5^1 - 1 = 4 = 4 \cdot 1$, which is divisible by 4.	*The problem asks you to label the base explicitly: do it.*
Inductive hypothesis. Let $k \geq 1$ be an integer and assume $4 \mid 5^k - 1$, that is, $5^k - 1 = 4a$ for some integer a.	*Turn "divisible" into an equation. This is what lets you compute with the IH.*
Inductive step. We show $4 \mid 5^{k+1} - 1$. We have $$5^{k+1} - 1 = 5 \cdot 5^k - 5 + 4 = 5\,(5^k - 1) + 4 \stackrel{\text{IH}}{=} 5 \cdot 4a + 4 = 4\,(5a + 1).$$ Since $5a + 1$ is an integer, $4 \mid 5^{k+1} - 1$.	*Add and subtract 5 to create $5(5^k - 1)$. The factor $5a + 1$ must be an integer: say so.*
Conclusion. By the principle of mathematical induction, $4 \mid 5^n - 1$ for every $n \in \mathbb{N}$. ■	

Remarks

If $0 \in \mathbb{N}$. Then also check $n = 0$: $5^0 - 1 = 0 = 4 \cdot 0$, so the base case $n = 0$ works and the same step covers all $n \geq 0$.
Why it is true, in one line. $5^n - 1 = (5 - 1)(5^{n-1} + 5^{n-2} + \cdots + 1)$. Induction is not the only proof, but it is the one the problem asks for.

2.6 Problem 5: $n! > 2^n$

Problem 5

Prove by induction that $n! > 2^n$ for $n \geq 4$.

How to think about it. Inequality shape. Going from k to $k+1$, the left side is multiplied by $k+1$ and the right side only by 2. Since $k+1 > 2$, the left side keeps winning. The base is 4 because $3! = 6 < 8 = 2^3$.

Model solution to Problem 5

Solution (what you write)	Commentary (what it does)
Statement. For every integer $n \geq 4$, $n! > 2^n$. We prove this by induction on n.	
Base case. For $n = 4$: $4! = 24$ and $2^4 = 16$, and $24 > 16$.	*Show both numbers.*
Inductive hypothesis. Let $k \geq 4$ be an integer and assume $k! > 2^k$.	
Inductive step. We show $(k+1)! > 2^{k+1}$. Using $(k+1)! = (k+1) \cdot k!$, $$(k+1)! = (k+1)\,k! \overset{\text{IH}}{>} (k+1)\,2^k > 2 \cdot 2^k = 2^{k+1}.$$ The first inequality uses the IH and $k+1 > 0$; the second uses $k+1 \geq 5 > 2$ (since $k \geq 4$).	*Justify each ">". Multiplying an inequality by $k+1$ keeps its direction because $k+1$ is positive. The second inequality is where $k \geq 4$ is used.*
Conclusion. By the principle of mathematical induction, $n! > 2^n$ for all $n \geq 4$. ■	

n	1	2	3	**4**	5	6
$n!$	1	2	6	**24**	120	720
2^n	2	4	8	**16**	32	64
$n! > 2^n$?	✗	✗	✗	✓	✓	✓

Table 4. Problem 5: the claim is false for $n \leq 3$, which is why the base case is $n = 4$.

2.7 Problem 6: $5 \mid 8^n - 3^n$

Problem 6

Prove by induction that for all $n > 1$, $8^n - 3^n$ is divisible by 5.

How to think about it. Divisibility shape, but with two powers. The trick is to add and subtract $3 \cdot 8^k$ (or $8 \cdot 3^k$) so that $8^k - 3^k$ appears: $8^{k+1} - 3^{k+1} = 8 \cdot 8^k - 3 \cdot 8^k + 3 \cdot 8^k - 3 \cdot 3^k = 5 \cdot 8^k + 3\,(8^k - 3^k)$.

About the range in the statement

The sheet says "$n > 1$", but the answer key starts the induction at $n = 1$. The statement is in fact true for all $n \geq 1$ ($n = 1$ gives $8 - 3 = 5$), and proving it for all $n \geq 1$ proves it in particular for all $n > 1$. So we prove the stronger statement. (Starting at $n = 2$ is also a correct proof of the statement as written: $64 - 9 = 55 = 5 \cdot 11$.)

Model solution to Problem 6

Solution (what you write)	**Commentary (what it does)**
Statement. For every integer $n \geq 1$, $5 \mid 8^n - 3^n$. (This includes all $n > 1$.) We prove it by induction on n.	*State which range you prove, and why it is enough.*
Base case. For $n = 1$: $8^1 - 3^1 = 5 = 5 \cdot 1$, which is divisible by 5.	
Inductive hypothesis. Let $k \geq 1$ and assume $5 \mid 8^k - 3^k$, that is, $8^k - 3^k = 5b$ for some integer b.	
Inductive step. We show $5 \mid 8^{k+1} - 3^{k+1}$. Adding and subtracting $3 \cdot 8^k$, $$\begin{aligned} 8^{k+1} - 3^{k+1} &= 8 \cdot 8^k - 3 \cdot 8^k + 3 \cdot 8^k - 3 \cdot 3^k \\ &= (8-3)\,8^k + 3\,(8^k - 3^k) \\ &\overset{\text{IH}}{=} 5 \cdot 8^k + 3 \cdot 5b \;=\; 5\,(8^k + 3b). \end{aligned}$$ Since $8^k + 3b$ is an integer, $5 \mid 8^{k+1} - 3^{k+1}$.	*The answer key stops at $5 \cdot 8^k + 3(8^k - 3^k)$. Finish the argument: say why that sum is a multiple of 5 (each term is), or factor out the 5 as here.*
Conclusion. By the principle of mathematical induction, $5 \mid 8^n - 3^n$ for all $n \geq 1$, and in particular for all $n > 1$. ■	

2.8 Problem 7: a geometric sum

Problem 7

Given a non-negative integer n, use mathematical induction to show that $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$.

How to think about it. Sum shape (Section 1.6): split off the last term 2^{k+1}. Note the base case is $n = 0$, because the statement is claimed for every non-negative integer.

Model solution to Problem 7

Solution (what you write)	**Commentary (what it does)**
Statement. For every integer $n \geq 0$, $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$. We prove this by induction on n.	
Base case. For $n = 0$: the left side is the single term $2^0 = 1$, and the right side is $2^{0+1} - 1 = 1$. So the statement holds.	*Base case $n = 0$, not $n = 1$: the range starts at 0.*
Inductive hypothesis. Let $k \geq 0$ and assume $2^0 + 2^1 + \cdots + 2^k = 2^{k+1} - 1$.	
Inductive step. We show $2^0 + 2^1 + \cdots + 2^{k+1} = 2^{k+2} - 1$. Splitting off the last term, $$\begin{aligned} 2^0 + \cdots + 2^k + 2^{k+1} &\overset{\text{IH}}{=} \left(2^{k+1} - 1\right) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1. \end{aligned}$$	*Write the goal with $k+1$ substituted everywhere: the exponent on the right becomes $(k+1)+1 = k+2$.*
Conclusion. By the principle of mathematical induction, the identity holds for every integer $n \geq 0$. ■	
Check. $n = 3$: $1 + 2 + 4 + 8 = 15 = 2^4 - 1$. ✓	

A binary view of Problem 7

Written in binary, the left side is a string of $n+1$ ones, and adding 1 carries all the way:

$$2^0 + 2^1 + \cdots + 2^n = (\underbrace{11\cdots1}_{n+1})_2, \qquad (\underbrace{11\cdots1}_{n+1})_2 + 1 = (1\underbrace{00\cdots0}_{n+1})_2 = 2^{n+1}.$$

Induction proves the identity; the binary picture explains it.

3 Mistakes, checklist and exercises

3.1 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **Multiplying route types instead of adding them.** In Problem 1(a), $2 \cdot 4 \cdot 3 = 24$ is wrong: a trip uses the direct roads *or* the roads through B. Multiply along a route, add over routes.
2. **Ignoring whether the containers are identical.** Identical pots: $\binom{7}{3}$; labelled pots: $P(7,3)$. Read the problem for words like "identical", "numbered", "labelled".
3. **"Minimum first, then the rest" for distinct objects.** It counts some selections several times (Problem 2(b): 4800 instead of 1200). List the exact compositions instead.
4. **Swapping n and r in stars and bars**, or using the non-negative formula when the variables must be positive. n is the number of variables, r the right-hand side; for $x_i \geq 1$ subtract the lower bounds first.
5. **No base case, or the wrong base case.** The base is the first n in the claimed range ($n = 0$ in Problem 7, $n = 4$ in Problem 5). Without a base case, a "proof" can prove false statements (spot the error 2).
6. **Assuming what you want to prove.** The IH is $P(k)$ for one arbitrary k. Writing "assume the statement holds for all n", or starting the step from the equation $P(k+1)$ and simplifying to something true, is circular (spot the error 3).
7. **Not using the IH, or not showing where.** If the IH does not appear in your step, the step is almost certainly wrong or incomplete. Mark it with "$\stackrel{\text{IH}}{=}$" or "by the inductive hypothesis".
8. **Stopping one line too early.** In a divisibility step, end with "$= d \cdot (\text{integer})$" and say the bracket is an integer. Every proof ends with the conclusion sentence.

Spot the error 1: Problem 2(b)

"Solution." The bouquet must contain at least 3 of each kind, so first choose 3 roses, 3 tulips and 3 lotuses: $\binom{6}{3}\binom{5}{3}\binom{4}{3} = 800$ ways. Then choose the 10th flower from the $15 - 9 = 6$ flowers that are left: 6 ways. Answer: $800 \cdot 6 = 4800$.
Where is the mistake? (Answer in Appendix A.) Hint: the correct answer is 1200. What is 4800/1200, and where does that number come from?

Spot the error 2: a "proof" with no base case

"Claim." For every $n \geq 1$, the number $n^2 + n + 1$ is even.
"Proof." Assume $k^2 + k + 1$ is even. Then $(k+1)^2 + (k+1) + 1 = (k^2 + k + 1) + 2(k+1)$, which is an even number plus an even number, hence even. By induction the claim holds for all $n \geq 1$.
Where is the mistake? (Answer in Appendix A.) Hint: compute $n^2 + n + 1$ for $n = 1$.

Spot the error 3: Problem 7 backwards

"Inductive step." We want $2^0 + \cdots + 2^{k+1} = 2^{k+2} - 1$. Subtracting 2^{k+1} from both sides gives $2^0 + \cdots + 2^k = 2^{k+2} - 2^{k+1} - 1 = 2^{k+1} - 1$, which is true by the inductive hypothesis. Hence the statement holds for $k + 1$.
Where is the mistake? (Answer in Appendix A.)

Spot the error 4: Problem 3(b)

"Solution." By stars and bars with $n = 32$ and $r = 4$, the number of solutions is $\binom{32+4-1}{4} = \binom{35}{4} = 52\,360$.

Where is the mistake? (Answer in Appendix A.)

3.2 Checklist before you hand in

Checklist

- ☐ **Counting:** I stated what is counted and decided order / repetition / identical objects *and containers*.
- ☐ For a map, I split the trips into **route types**, multiplied along each type and added over the types.
- ☐ For "at least" with distinct objects, I listed the **exact compositions** and explained why the list is complete.
- ☐ For identical objects into boxes, or integer solutions, I used **stars and bars** with the right n and r, after subtracting any lower bounds.
- ☐ **Induction:** I wrote $P(n)$ and its range, and the **base case** is the first value in that range, checked numerically.
- ☐ The **inductive hypothesis** is assumed for one arbitrary $k \geq n_0$ and written out in full (as an equation for divisibility).
- ☐ The **goal** $P(k+1)$ is written out, the step goes from one side of it to the other, and the use of the IH is marked.
- ☐ I ended with the **conclusion** sentence, and every counting answer is **boxed** (expression and number).

3.3 Exercises

Write each solution in full, using the templates.

E1. In Problem 1, how many round trips A → C → A use *no road* of the outbound trip on the way back?

E2. With the florist's 6 roses, 5 tulips and 4 lotuses (all different), how many bouquets of 8 flowers contain at least 2 of each kind?

E3. How many integer solutions does $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ have with all $x_i \geq 0$, $x_1 \geq 3$ and $x_5 \leq 2$? (Hint: for the upper bound, use the complement.)

E4. Twelve identical candies are handed out to 4 children. In how many ways can this be done (a) with no restriction; (b) if every child gets at least 2?

E5. How many non-negative integer solutions does the *inequality* $x_1 + x_2 + x_3 + x_4 \leq 32$ have? (Hint: introduce a fifth variable $x_5 = 32 - (x_1 + \cdots + x_4) \geq 0$.)

E6. Prove by induction that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for every $n \geq 1$.

E7. Prove by induction that $3 \mid n^3 + 2n$ for every $n \geq 0$.

E8. Prove by induction that $3^n \geq 2^n + n$ for every $n \geq 1$.

E9. Prove by induction that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for every $n \geq 1$.

E10. **Find the flaw.** "Claim: in every set of $n \geq 1$ horses, all horses have the same colour. Base case: one horse has one colour. Step: given $k+1$ horses, the first k have the same colour by the IH, and so do the last k; the two groups overlap, so all $k+1$ have the same colour." Which step fails, and for which k?

A Answers to the "spot the error" puzzles

Spot the error 1 (minimum first). The procedure does not produce each bouquet exactly once. Take a bouquet with 4 roses $\rho_1, \rho_2, \rho_3, \rho_4$ (and 3 tulips, 3 lotuses). It is produced when the "first three roses" are $\{\rho_2, \rho_3, \rho_4\}$ and the 10th flower is ρ_1, and also when they are $\{\rho_1, \rho_3, \rho_4\}$ with 10th flower ρ_2, and so on: 4 times, once for each choice of which of its 4 roses is "the extra one". Every valid bouquet has exactly one kind with 4 flowers, so every bouquet is counted exactly 4 times, and indeed $4800/4 = 1200$. (Here the overcount happens to be uniform, so dividing works; in general it is not, which is why listing the compositions is the safe method.)

Spot the error 2 (no base case). The inductive step is correct, but the base case fails: $1^2 + 1 + 1 = 3$ is odd. In fact $n^2 + n = n(n+1)$ is always even, so $n^2 + n + 1$ is *always odd* and the claim is false for every n. The step only shows "if it were true for k, it would be true for $k+1$"; without a first domino, nothing falls.

Spot the error 3 (working backwards). The argument starts from the goal $P(k+1)$, i.e. it assumes what it wants to prove, and derives a true statement. That is not valid logic: from $-1 = 1$ one can derive the true statement $1 = 1$ by squaring, but $-1 \neq 1$. Here each step happens to be reversible, so the repair is easy: start from the IH and *add* 2^{k+1} to both sides, as in the model solution. Always go from what you know to what you want.

Spot the error 4 (swapped parameters). In $\binom{n+r-1}{r}$, n is the number of variables (4) and r is the right-hand side (32). The "solution" counts the ways to put 4 stars into 32 boxes. Correct: $\binom{4+32-1}{32} = \binom{35}{3} = 6545$. (Self-test: the bottom of $\binom{n+r-1}{n-1}$ is the number of bars, 3.)