Tutorial 1: The Basic Counting Toolbox

Sum and product rules, permutations, combinations, and how to write a counting argument

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Decide, for any counting question, whether *order matters*, whether *repetition* is allowed, and whether some objects are *indistinguishable*, and pick the matching tool.
2. Use the rules of sum and product, $P(n,r)$, $\binom{n}{r}$, and the formula $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$ for arrangements with repeated objects.
3. Count "at least one" and "does not contain" questions with the complement and with inclusion-exclusion, and "no two adjacent" questions with the gap method.
4. Write a counting solution in the standard format: **Statement** $\to$ **Solution** (set-up, then a step-by-step count naming the rule used at each step) $\to$ **Answer**, and give a *combinatorial argument* for why an expression is an integer.

Sections 1.1 to 1.7 review the tools with a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 solves all four tutorial problems in full. Section 3 collects the classic mistakes, a checklist and extra exercises.

Contents

1 The toolbox

Every problem in this tutorial is solved by combining a handful of tools. Before the problems, here is each tool explained the same way: the idea in plain words, a small example, the precise statement, a bigger example, and a recap.

1.1 The rule of sum and the rule of product

The intuition

Two words tell you which rule to use.

- **"or" means add.** If you must pick *one* thing, and it is either of the first kind (which comes in m varieties) *or* of the second kind (n varieties), you have $m + n$ options.
- **"and then" means multiply.** If you must do a first step *and then* a second step, and the first step has m options and the second always has n options, the two steps together have $m \cdot n$ outcomes.

Simple example: two classes

There are 90 students in COMP 339 and 20 students in MATH 339 (no student is in both).

- Choose *one* student from the two classes together: $90 + 20 = 110$ ways [rule of sum].
- Choose one representative from *each* class: $90 \cdot 20 = 1800$ ways [rule of product].

Rule of sum and rule of product

Rule of sum. If a task can be done in one of m ways or in one of n ways, and *none of the m ways is also one of the n ways*, then the task can be done in $m + n$ ways. (In set language: if A and B are disjoint finite sets, $|A \cup B| = |A| + |B|$.)

Rule of product. If a procedure consists of a first step and a second step, the first step can be done in m ways, and *for each of these* the second step can be done in n ways, then the procedure can be carried out in $m \cdot n$ ways. For k steps with $n_1, n_2, \ldots, n_k$ options the count is $n_1 n_2 \cdots n_k$.

The fine print. The rule of product needs the *number* of options at each step to be the same no matter what was chosen earlier (the options themselves may differ). If the number changes depending on earlier choices, either reorder the steps or split into cases and use the rule of sum.

The slot picture. Most product-rule counts are "fill in the slots" counts: draw one box per position, write above each box how many symbols may go there, and multiply.

Comprehensive example: 4-digit decimal numbers

How many 4-digit strings of decimal digits are there (a) with repeated digits allowed, and a leading 0 allowed; (b) with no repeated digits; (c) with no repeated digits and no leading 0?

Solution. (a) Each of the four slots takes any of 10 digits: $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10\,000$.

(b) The first slot has 10 options; whichever digit was used, the second slot has 9 options left, then 8, then 7: $10 \cdot 9 \cdot 8 \cdot 7 = 5040$.

(c) Fill the *most restricted* slot first. The first digit cannot be 0: 9 options. The second digit can be anything except the first digit (so 0 is allowed now): 9 options. Then 8, then 7: $9 \cdot 9 \cdot 8 \cdot 7 = 4536$.

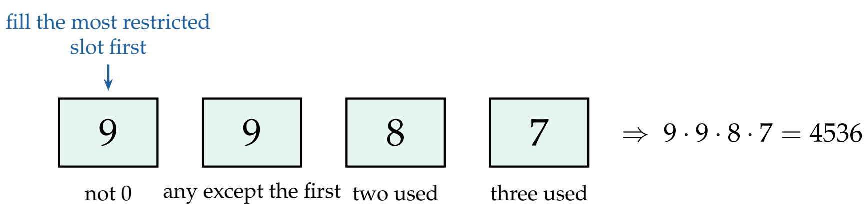


Figure 1. The slot picture for part (c). Each box shows how many digits may go in that position.

Why the order of the slots matters

Try (c) by filling the *last* slot first: 10 options, then 9, then 8, and now the first slot has “7 options, minus one if 0 has not been used yet”. The number of options depends on earlier choices, so the rule of product does not apply as it stands. Filling the restricted slot first avoids this. (The other repair is to split into cases: “0 appears among the last three digits” or “it does not”, and add.)

Sum and product in one breath

- “Either … or …” with no overlap: add. “First … and then …”: multiply.
- Draw the slots, write the number of options above each, multiply.
- Fill the most restricted slot first; if the number of options still depends on earlier choices, split into cases.

1.2 Permutations: arranging distinct objects in a row

The intuition

A *permutation* is an arrangement in a line, so **order matters**: ART, TAR and RAT are three different arrangements of the same three letters. Arranging is a product-rule count in disguise: the first position can be any of the n objects, the next any of the remaining $n-1$, and so on.

Simple example: anagrams

The letters of MATH can be arranged in $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ ways, and the letters of CHAPTER in $7! = 5040$ ways. If we only want 4-letter "words" made from the 7 letters of CHAPTER (no letter used twice), the count is $7 \cdot 6 \cdot 5 \cdot 4 = 840$, which is $\frac{7!}{3!}$.

Factorials and $P(n, r)$

For an integer $n \geq 1$, $n! = n \cdot (n-1) \cdots 2 \cdot 1$, and by convention $0! = 1$.
If n objects are all different and $0 \leq r \leq n$, the number of ways to arrange r of them in a row (an *ordered* selection *without* repetition) is

$$P(n,r) \;=\; n\,(n-1)\cdots(n-r+1) \;=\; \frac{n!}{(n-r)!}.$$

In particular $P(n,n) = n!$ is the number of ways to arrange all n objects.
If repetition *is* allowed (each of r slots may hold any of n symbols), the count is n^r.

Comprehensive example: fruit in lunch boxes

One banana, one apple and one orange are to be put into 10 numbered lunch boxes. (a) At most one piece of fruit per box: the banana has 10 choices of box, then the apple 9, then the orange 8, so $10 \cdot 9 \cdot 8 = P(10,3) = 720$. (b) Boxes may receive more than one piece: $10 \cdot 10 \cdot 10 = 10^3 = 1000$. Notice what is being arranged here: it is the *boxes* that are assigned to the three (distinct) fruits, one after another. "Distinct objects, order matters, no repetition" is the signature of $P(n,r)$.

Permutations in one breath

- Distinct objects, order matters, no repetition: $P(n,r) = \frac{n!}{(n-r)!}$; all of them: $n!$.
- Order matters, repetition allowed: n^r.
- Both are just the rule of product written compactly.

1.3 Arrangements with repeated (indistinguishable) objects

The intuition

The word LECTURE has two E's. If the E's were different, say E_1 and E_2, there would be 7! arrangements. But swapping E_1 and E_2 produces the *same word*, so every real word has been counted exactly twice. The number of distinct words is therefore $7!/2$. In general: **first pretend the copies are different, then divide by the number of ways the copies can be shuffled among themselves.**

Simple example: BANANAS

BANANAS has 7 letters: three A's, two N's, one B, one S. Pretending all seven are different gives 7! arrangements. The three A's can be shuffled among their positions in 3! ways and the two N's in 2! ways without changing the word, so each word is counted $3! \cdot 2!$ times. Hence

$$\#\text{arrangements of BANANAS} = \frac{7!}{3!\,2!} = \frac{5040}{12} = 420.$$

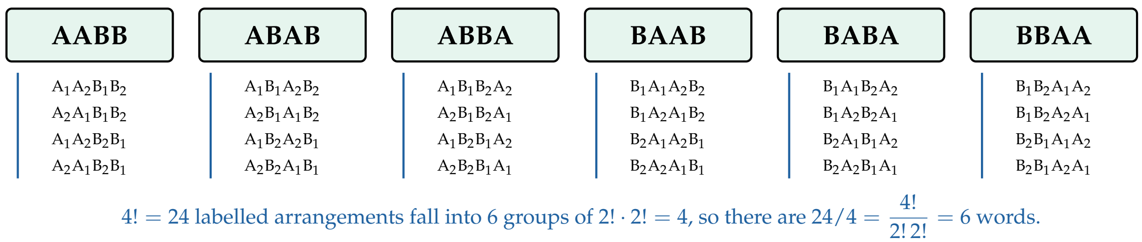


Figure 2. Why we divide: the labelled arrangements of $A_1A_2B_1B_2$ *collapse into groups of equal size, one group per word made of A, A, B, B.*

Arrangements of a multiset

Suppose there are n objects of r types: n_1 identical objects of type 1, n_2 of type 2, …, n_r of type r, with $n_1 + n_2 + \cdots + n_r = n$. The number of distinct linear arrangements of all n objects is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

Key consequence. This expression *counts something*, so it is always a positive integer, whatever the n_i are. (Section 1.7 and Problem 3 build on exactly this observation.)

Comprehensive example: assigning TAs to time slots

Six TAs are to be assigned to three one-hour slots A, B, C, two TAs per slot. How many assignments are there?

Solution 1 (choose slot by slot). Choose the two TAs for slot A: $\binom{6}{2}$ ways; then two of the remaining four for slot B: $\binom{4}{2}$; the last two take slot C: $\binom{2}{2}$. By the rule of product, $\binom{6}{2}\binom{4}{2}\binom{2}{2} = 15 \cdot 6 \cdot 1 = 90$.

Solution 2 (a word with repeated letters). Write the TAs in a fixed order $1, \ldots, 6$ and under each TA write the letter of their slot. An assignment is exactly a 6-letter word with two A's, two B's and two C's, and there are $\frac{6!}{2!\,2!\,2!} = \frac{720}{8} = 90$ such words. Same answer, as it must be.

Repeated objects in one breath

- Pretend the copies are different ($n!$), then divide by $n_i!$ for each group of n_i identical objects.
- $\frac{n!}{n_1!\cdots n_r!}$ is a count, hence an integer.
- Many "assign each object a label" problems are secretly "arrange a word with repeated letters".

1.4 Combinations: choosing without order

The intuition

A committee of $\{\text{Ann, Bob, Cy}\}$ is the same committee as $\{\text{Cy, Ann, Bob}\}$: when you *choose* a set, order does not matter. If you first count ordered selections, $P(n,r)$, every unordered selection of r objects has been counted $r!$ times (once per ordering), so the number of unordered selections is $P(n,r)/r!$.

Simple example: three identical bananas

Put 3 identical bananas into 10 numbered boxes, at most one per box. If the bananas were distinct the answer would be $P(10,3) = 720$. Since they are identical, the $3! = 6$ orders in which a given set of three boxes can be filled all give the same outcome, so the answer is $720/6 = 120$. We are simply *choosing which* 3 *boxes* get a banana.

Combinations, $\binom{n}{r}$

For integers $0 \le r \le n$, the number of ways to choose an (unordered) subset of r objects from n distinct objects is

$$\binom{n}{r} = C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!},$$

read "n choose r". Useful facts: $\binom{n}{0} = \binom{n}{n} = 1$, $\binom{n}{1} = n$, and $\binom{n}{r} = \binom{n}{n-r}$ (choosing who is *in* is the same as choosing who is *out*).

The string picture. A binary string of length n with exactly r ones is determined by *which r positions* hold the ones. So there are $\binom{n}{r}$ such strings, and summing over r gives all 2^n binary strings: $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

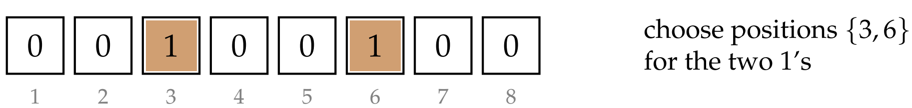


Figure 3. A byte with exactly two 1's is the same thing as a choice of 2 positions out of 8: there are $\binom{8}{2} = 28$ of them. This is Problem 4(a).

Comprehensive example: committees

(a) From 80 students, how many committees of 3 can be formed? Order does not matter: $\binom{80}{3} = \dfrac{80 \cdot 79 \cdot 78}{3!} = \dfrac{492\,960}{6} = 82\,160$.

(b) From 130 first-year and 181 second-year students, how many ways are there to pick a pair consisting of one first-year and one second-year student? Choose the first-year student *and then* the second-year student: $\binom{130}{1}\binom{181}{1} = 130 \cdot 181 = 23\,530$.

(c) How many bytes contain *at most* one 1? Either no 1 ($\binom{8}{0} = 1$ byte) *or* exactly one 1 ($\binom{8}{1} = 8$ bytes): $1 + 8 = 9$ by the rule of sum.

Combinations in one breath

- Choosing a set (order irrelevant, no repetition): $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

- $\binom{n}{r}$ also counts binary strings of length n with r ones, and r-element subsets of an n-element set.
- Ask "does swapping two chosen things give a new outcome?" If no, it is a combination.

1.5 Counting the complement, and inclusion-exclusion

The intuition

Questions with "at least one" or "does not contain" are usually easier *backwards*: count everything, then subtract the outcomes you do not want. "At least one 0" is the opposite of "no 0 at all", and "no 0 at all" is a plain slot count.
When there are two unwanted properties, subtracting each of them removes the outcomes having *both* twice, so those must be added back once. That correction is called *inclusion-exclusion*.

Simple example: 5-digit strings with a 0

Consider all 10^5 strings of five decimal digits (leading 0 allowed).

- Containing at least one 0: total minus those with no 0: $10^5 - 9^5 = 100\,000 - 59\,049 = 40\,951$.
- Containing at most one 0: no 0 (9^5) or exactly one 0 (choose its position, 5 ways, fill the rest from nine digits, 9^4): $9^5 + 5 \cdot 9^4 = 59\,049 + 32\,805 = 91\,854$ [rule of sum].

Complement and inclusion-exclusion

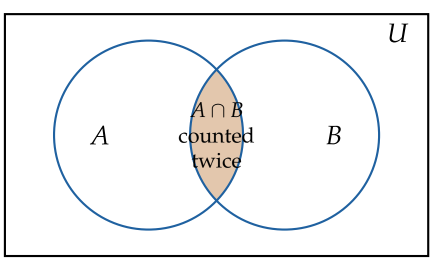
Let U be the finite set of all outcomes and $A \subseteq U$.
Complement. $|A| = |U| - |U \setminus A|$: the number of outcomes *with* a property equals the total minus the number *without* it.
Inclusion-exclusion (two sets). For finite sets A and B,

$$|A \cup B| \;=\; |A| + |B| - |A \cap B|.$$

Inclusion-exclusion (three sets).

$$|A \cup B \cup C| \;=\; |A| + |B| + |C| \;-\; |A \cap B| - |A \cap C| - |B \cap C| \;+\; |A \cap B \cap C|.$$

Typical use: A, B, C are the *bad* outcomes ("contains no 0", "contains OK at position 1", ...), and the answer is $|U| - |A \cup B \cup C|$.



$|A| + |B|$ counts the shaded part twice, so $|A \cup B| = |A| + |B| - |A \cap B|$.

Figure 4. Inclusion-exclusion for two sets: the overlap is added back once.

Comprehensive example: a 0 and a 1

How many 5-digit strings (leading 0 allowed) contain at least one 0 *and* at least one 1?
Solution. Let U be all 10^5 strings, A the strings with no 0, and B the strings with no 1. A string is *bad* if it lacks a 0 or lacks a 1, i.e. belongs to $A \cup B$. Now $|A| = 9^5$ (five slots, nine digits each), $|B| = 9^5$, and $|A \cap B| = 8^5$ (no 0 and no 1: eight digits per slot). By inclusion-exclusion, $|A \cup B| = 9^5 + 9^5 - 8^5$, and the answer is

$$10^5 - 9^5 - 9^5 + 8^5 \;=\; 100\,000 - 59\,049 - 59\,049 + 32\,768 \;=\; 14\,670.$$

Complement and inclusion-exclusion in one breath

- "At least one" or "must contain": count the complement ("none"), subtract from the total.
- Two unwanted properties: subtract each, add back the overlap. Three: subtract singles, add pairs, subtract the triple.
- Always say what U, A, B are before computing.

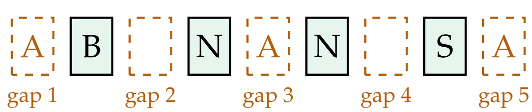
1.6 The gap method for "no two adjacent"

The intuition

To keep some objects (say the A's) apart, do *not* place them first. Place *everything else* first; the placed objects create *gaps* (one before each object and one after the last). Now drop the A's into the gaps, at most one A per gap. Two A's in different gaps are automatically separated by at least one other object.

Simple example: BANANAS with no two A's adjacent

Step 1: arrange the letters other than A, namely B, N, N, S: $\dfrac{4!}{2!} = 12$ ways (two identical N's).
Step 2: these four letters create 5 gaps: _ B _ N _ N _ S _. Step 3: choose 3 of the 5 gaps for the (identical) A's: $\binom{5}{3} = 10$ ways. By the rule of product: $12 \cdot 10 = 120$ arrangements.



4 placed letters $\Rightarrow$ 5 gaps;
choose 3 gaps for the A's: $\binom{5}{3}$.
Here: gaps $1, 3, 5$ give ABNANSA.

Figure 5. The gap method. Solid boxes: the letters placed first. Dashed boxes: the gaps. At most one A per gap guarantees no two A's touch.

The gap method (procedure)

To count arrangements in which no two of s "special" objects are adjacent:

1. Arrange the m *other* objects in a row (use $m!$, or $\frac{m!}{n_1! \cdots}$ if some are identical).
2. They create $m+1$ gaps (before the first, between consecutive ones, after the last).
3. Place the special objects, at most one per gap: $\binom{m+1}{s}$ ways if they are identical, and $P(m+1, s) = \binom{m+1}{s}\, s!$ ways if they are all different (choose the gaps, then order the objects in them).
4. Multiply (rule of product).

The method needs $s \le m+1$; if $s > m+1$ the count is 0, since there are not enough gaps.

Comprehensive example: labelled special objects

How many arrangements of the letters B, N, N, S and three *different* symbols α, β, γ have no two of α, β, γ adjacent?
Arrange B, N, N, S: $4!/2! = 12$. Five gaps; choose three of them and decide which symbol goes where: $P(5,3) = 5 \cdot 4 \cdot 3 = 60$ (equivalently $\binom{5}{3} \cdot 3! = 10 \cdot 6$). Answer: $12 \cdot 60 = 720$. Compare with 120 for identical A's: labelling the special objects multiplies the count by 3!.

The gap method in one breath

- Place the others, count the gaps (m objects give $m+1$ gaps), choose gaps for the special objects, multiply.
- Identical special objects: $\binom{m+1}{s}$. Distinct ones: $P(m+1, s)$.
- Do not "glue" objects together to count the complement unless there are exactly two special objects (see Section 3).

1.7 Combinatorial arguments

The intuition

A *combinatorial argument* proves a fact about numbers by *counting something*. Two classic uses:
- To show an expression is an **integer**: exhibit a finite set whose size is that expression. Sizes of finite sets are integers, so you are done, with no algebra at all.
- To show two expressions are **equal**: exhibit one set and count it in two different ways.

Simple example: $\binom{n}{r}$ is an integer, and $\binom{n}{r} = \binom{n}{n-r}$

The number $\dfrac{n!}{r!\,(n-r)!}$ looks like a fraction, but it equals the number of r-element subsets of an n-element set, so it is an integer. Moreover, choosing which r elements are *in* a subset is the same as choosing which $n-r$ elements are *out*, so the same set of subsets is counted by $\binom{n}{n-r}$. Hence $\binom{n}{r} = \binom{n}{n-r}$.

What a combinatorial argument must contain

(i) **Name a concrete finite set.** "Consider the set S of all arrangements of the symbols ..."
(ii) **Show that $|S|$ equals the expression.** Use a rule you already know (product rule, multiset formula, $\binom{n}{r}$, ...), and say why every element of S is counted exactly once.
(iii) **Conclude.** "Since the expression is the number of elements of a finite set, it is a non-negative integer." (For an equality: "Both expressions count $|S|$, hence they are equal.")

Comprehensive example: $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

Statement. For every integer $n \geq 0$, $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Solution. Let S be the set of all binary strings of length n.
Count 1. Each of the n positions holds 0 or 1, so by the rule of product $|S| = 2^n$.
Count 2. Sort the strings by their number of ones. A string with exactly k ones is determined by the set of k positions holding the ones, so there are $\binom{n}{k}$ such strings. The values $k = 0, 1, \ldots, n$ are mutually exclusive and cover all of S, so by the rule of sum $|S| = \sum_{k=0}^{n} \binom{n}{k}$.
Conclusion. Both expressions equal $|S|$, so they are equal. ■
(This also follows from the binomial theorem $(x+y)^n = \sum_k \binom{n}{k} x^{n-k} y^k$ with $x = y = 1$; the combinatorial argument explains *why* it is true.)

Combinatorial arguments in one breath

- Integer? Find a finite set of that size. Equal? Count one set two ways.
- Always name the set explicitly and justify the count with a known rule.
- The multiset formula $\dfrac{n!}{n_1! \cdots n_r!}$ is the workhorse for "this quotient of factorials is an integer".

The toolbox at a glance

Situation	Count	Signal words
Pick one thing from disjoint kinds	$m + n$	"either … or …"
Do step 1 and then step 2	$m \cdot n$	"and then", slots, digits, letters
Arrange r of n distinct objects, no repetition	$P(n, r) = \frac{n!}{(n-r)!}$	"arrange", "in a row", "ordered"
Fill r slots from n symbols, repetition allowed	n^r	"strings", "digits may repeat"
Arrange all n objects, n_i identical of type i	$\frac{n!}{n_1! \cdots n_r!}$	"anagrams", identical letters
Choose r of n distinct objects, order irrelevant	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	"committee", "subset", "positions of the ones"
"At least one", "must contain"	total − "none"	complement
Two or three unwanted properties	inclusion-exclusion	"does not contain … or …"
No two special objects adjacent	others first, then gaps: $\binom{m+1}{s}$ or $P(m+1, s)$	"not adjacent", "separated"
Show an expression is an integer	find a set it counts	"combinatorial argument"

2 The tutorial problems, worked in full

2.1 How to write a counting solution

A counting answer is not a number; it is an *argument* that ends in a number. Graders look for three things: what exactly is being counted, which rule justifies each factor, and a clearly marked final answer. Use the same skeleton every time.

Writing template for a counting solution

Statement. *[Restate the question precisely: what objects, what makes one "valid".]*
Solution. *Set-up. [Say what you are counting and settle the three questions: does order matter? is repetition allowed? are some objects identical? If you count the complement, say so: "We count the strings with no* 0 *and subtract."]*
Count. [Go step by step. After each factor name the rule: "by the rule of product", "since the e's are indistinguishable", "choose 3 *of the* 5 *gaps". If a step has cases, list them and add.]*
Answer. *[Give the expression and, when it is reasonable, its numerical value:* $4! \cdot \binom{5}{3} = 240$*.]*
Check (optional but recommended). [A tiny case you can list by hand, or a second method giving the same number.]

Phrasebook: sentences that belong in a counting argument

Where	What to write
Setting up	"We count the number of …" "Order matters here because …" "The five e's are indistinguishable, so …" "Equivalently, we choose the positions of …"
Product rule	"There are m choices for …, and for each of these, n choices for …; by the rule of product …"
Sum rule	"Exactly one of the following cases occurs: …. Adding, …"
Repeated objects	"Pretending the copies are distinct gives $n!$; each arrangement is counted $n_1!\,n_2!$ times, so …"
Complement	"It is easier to count the strings that contain *no* … and subtract from the total."
Inclusion-exclusion	"Let A_1 be the set of … and A_2 the set of …. Then $\|A_1 \cup A_2\| = \|A_1\| + \|A_2\| - \|A_1 \cap A_2\|$ …"
Gap method	"First arrange the remaining m symbols; they create $m+1$ gaps. Choose … of the gaps for …"
Combinatorial argument	"Consider the set of all …. Its size is … by …. Since this is the size of a finite set, it is an integer."
Finishing	"Therefore the number of … is …" "Answer: …" "As a check, for $n = 3$ the list is …"

2.2 Problem 1: keeping the e's apart

Problem 1

a) In how many ways can the symbols $a, b, c, d, e, e, e, e, e$ be arranged so that no e is adjacent to another e?

b) Suppose the e's from part a) are now labelled: the symbols are a, b, c, d and e_1, e_2, e_3, e_4, e_5. In how many ways can these symbols be arranged so that no labelled e is adjacent to another one?

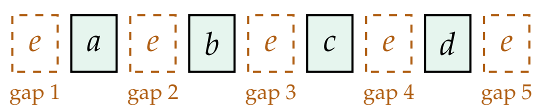
c) Suppose we have the symbols $a_1, a_2, \ldots, a_n$ and $n+1$ identical e's. In how many ways can these symbols be arranged so that no e is adjacent to another e?

d) Suppose the e's from part c) are now labelled. In how many ways can these symbols be arranged so that no labelled e is adjacent to another one?

How to think about it. "No two e's adjacent" is the signal for the gap method: place a, b, c, d first, then drop the e's into the gaps. Four placed letters create five gaps, and there are five e's, so *every* gap must receive exactly one e. The arrangement is forced to look like

$$e \,\square\, e \,\square\, e \,\square\, e \,\square\, e$$

with a, b, c, d filling the four boxes in some order. That observation solves all four parts.



4 letters $\Rightarrow$ 5 gaps, and 5 e's:
every gap gets exactly one e.
Only the order of a, b, c, d is free.

Figure 6. Problem 1(a): the five e's must occupy all five gaps, so the pattern $e \square e \square e \square e \square e$ is forced.

Model solution to Problem 1(a)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the arrangements of a, b, c, d and five identical e's in which no two e's are adjacent.	*Say what the objects are and which ones are identical.*
Solution. We use the gap method. First arrange the four distinct letters a, b, c, d in a row: $4!$ ways. These four letters create 5 gaps (before a, between consecutive letters, after the last letter). To avoid adjacent e's, each gap may hold at most one e. Since there are exactly 5 identical e's and 5 gaps, every gap receives exactly one e, which can be done in $\binom{5}{5} = 1$ way. By the rule of product the number of arrangements is $4! \cdot 1$.	*Name the method. Count the "others" first, then the gaps, then the placement. Writing $\binom{5}{5} = 1$ makes the structure visible even though the factor is 1.*
Answer. $\boxed{4! = 24}$.	*Box the answer; give the number.*
Check. Every valid arrangement has the form $e \square e \square e \square e \square e$ with a, b, c, d in the boxes, and there are $4!$ ways to fill the boxes. Same count.	*A second method that gives the same number is the best possible check.*

Model solution to Problem 1(b)

Solution (what you write)	Commentary (what it does)
Statement. Same as (a), but the e's are now $e_1, \ldots, e_5$, all different.	
Solution. Arrange a, b, c, d: $4!$ ways, creating 5 gaps. Each gap must receive exactly one e as before, but now the five *distinct* e's can be distributed into the five gaps in $5!$ ways (choose which e goes in gap 1, which in gap 2, and so on). By the rule of product the count is $4! \cdot 5!$.	*The only change from (a) is the last factor:* $\binom{5}{5} = 1$ *becomes* $P(5,5) = 5!$. *Labelling the special objects multiplies by* $5!$.
Answer. $\boxed{5! \cdot 4! = 120 \cdot 24 = 2880}$.	

Model solution to Problem 1(c)

Solution (what you write)	Commentary (what it does)
Statement. Count the arrangements of the distinct symbols $a_1, \ldots, a_n$ together with $n+1$ identical e's in which no two e's are adjacent.	
Solution. Arrange the n distinct symbols $a_1, \ldots, a_n$: $n!$ ways. They create $n+1$ gaps. There are $n+1$ identical e's and each gap may hold at most one, so every gap receives exactly one e: $\binom{n+1}{n+1} = 1$ way. By the rule of product the number of arrangements is $n! \cdot 1$.	*Exactly the argument of (a) with* 4 *replaced by* n. *The pattern is again forced:* $e \square e \square \cdots \square e$ *with the* a_i*'s in the boxes.*
Answer. $\boxed{n!}$.	
Check. $n = 4$ gives $4! = 24$, which is part (a). ✓	*Plug in the small case you already know.*

Model solution to Problem 1(d)

Solution (what you write)	Commentary (what it does)
Statement. Same as (c), but the $n+1$ e's are labelled $e_1, \ldots, e_{n+1}$.	
Solution. Arrange $a_1, \ldots, a_n$: $n!$ ways, creating $n+1$ gaps, each of which must receive exactly one e. The $n+1$ distinct e's can be assigned to the $n+1$ gaps in $(n+1)!$ ways. By the rule of product the count is $n!\,(n+1)!$.	
Answer. $\boxed{n!\,(n+1)!}$.	
Check. $n = 4$ gives $4! \cdot 5! = 2880$, which is part (b). ✓	

What if there were fewer e's?

The problem is "tight" because the number of e's equals the number of gaps. With only $s \leq 5$ identical e's and the letters a, b, c, d, the same method gives $4! \cdot \binom{5}{s}$ arrangements (for example $s = 3$ gives $24 \cdot 10 = 240$), and with s labelled e's it gives $4! \cdot P(5, s)$. With 6 or more e's there are not enough gaps, and the answer is 0.

2.3 Problem 2: course names

Problem 2

A course name has the format *four letters + three-digit number*, for example COMP 339.

a) In how many ways can one rearrange the course name COMP 339 such that it is a valid course name (four letters C, O, M, P and three numbers 3 and 9, no repetitions allowed, except two 3s)?

b) In how many ways can one rearrange the course name COMP 339 *with* letter and number repetitions such that it is a valid course name (four letters from C, O, M, P and three numbers from 3 and 9)?

c) What is the number of all COMP courses in the format COMP + three-digit number (the number cannot start with a 0) that do NOT contain 3 or 9?

d) What is the number of all COMP courses in the format COMP + three-digit number (the number cannot start with a 0) such that the sum of the digits is even?

e) What is the number of all possible courses in the format four letters + three-digit number (the number cannot start with a 0)?

f) What is the number of all possible courses in the format four letters + 339 that do NOT contain the substring "OK"?

How to think about it. Every part is a slot problem: four letter slots followed by three digit slots. The parts differ only in which symbols may go in each slot and whether a symbol may be reused. Draw the slots, decide the options per slot, and multiply; when a condition talks about "does not contain", switch to the complement.

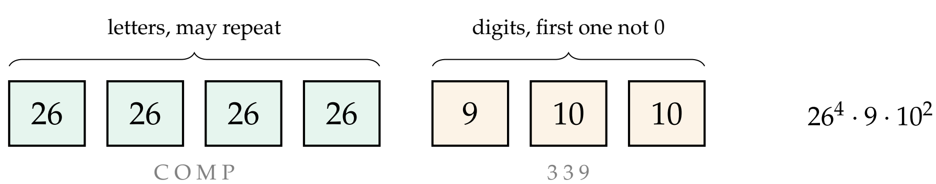


Figure 7. The slot picture for part (e). Parts (a) to (d) and (f) change the number of options in some slots.

Model solution to Problem 2(a)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the valid course names obtained by rearranging the symbols of COMP 339: the four letter slots hold C, O, M, P each exactly once, and the three digit slots hold the digits $3, 3, 9$ (each digit used exactly as often as in 339).	*Each symbol is used as often as it appears in COMP 339 (that is the meaning of "no repetitions except two 3s"). Letters stay in the letter slots and digits in the digit slots, otherwise the name is invalid.*
Solution. Arrange the four distinct letters C, O, M, P in the four letter slots: 4! ways. Arrange the digits $3, 3, 9$ in the three digit slots: the two 3's are identical, so there are $\frac{3!}{2!} = 3$ ways (equivalently, choose which of the 3 slots holds the 9: $\binom{3}{1} = 3$). The letter arrangement and the digit arrangement are chosen independently, so by the rule of product the count is $4! \cdot 3$.	*Two independent sub-arrangements, multiplied. The digit part is a multiset arrangement (Section 1.3) or, equivalently, a choice of one slot (Section 1.4).*

Answer. $\boxed{4! \cdot \binom{3}{1} = 24 \cdot 3 = 72}$.

Check. The digit blocks are exactly 339, 393, 933: three of them. ✓

Model solution to Problem 2(b)

Solution (what you write)	Commentary (what it does)
Statement. Count the course names whose four letter slots each hold one of C, O, M, P and whose three digit slots each hold 3 or 9, with repetitions allowed.	
Solution. Each of the 4 letter slots can be filled with any of the 4 letters, independently: 4^4 ways. Each of the 3 digit slots can be filled with either of the 2 digits: 2^3 ways. By the rule of product the count is $4^4 \cdot 2^3$.	*Repetition allowed and order matters: n^r for each block.*
Answer. $\boxed{4^4 \cdot 2^3 = 256 \cdot 8 = 2048}$.	

Model solution to Problem 2(c)

Solution (what you write)	Commentary (what it does)
Statement. Count the three-digit numbers (first digit not 0) in which neither 3 nor 9 appears; the letters are fixed as COMP.	*The letter block is fixed, so it contributes a factor of 1 and we only count the number.*
Solution. The allowed digits are $\{0, 1, 2, 4, 5, 6, 7, 8\}$, eight in total. The first digit cannot be 0, so it has $8 - 1 = 7$ options. The second and third digits may be any of the 8 allowed digits (repetition is allowed and 0 is fine here). By the rule of product the count is $7 \cdot 8 \cdot 8$.	*Remove the forbidden digits first, then apply the leading-digit restriction to the first slot only.*
Answer. $\boxed{7 \cdot 8^2 = 448}$.	

Model solution to Problem 2(d)

Solution (what you write)	Commentary (what it does)
Statement. Count the three-digit numbers $d_1 d_2 d_3$ with $d_1 \neq 0$ such that $d_1 + d_2 + d_3$ is even.	
Solution. Choose d_1: 9 options (1 to 9). Choose d_2: 10 options. Now $d_1 + d_2$ is some fixed integer. If $d_1 + d_2$ is even, the sum is even exactly when $d_3 \in \{0, 2, 4, 6, 8\}$; if $d_1 + d_2$ is odd, exactly when $d_3 \in \{1, 3, 5, 7, 9\}$. In either case there are exactly 5 options for d_3. The *number* of options for d_3 does not depend on the earlier choices, so the rule of product applies: $9 \cdot 10 \cdot 5$.	*The key sentence is "in either case there are exactly 5 options". That is what licenses the rule of product even though* which *five digits are allowed depends on $d_1 + d_2$.*
Answer. $\boxed{9 \cdot 10 \cdot 5 = 450}$, i.e. exactly half of the 900 three-digit numbers.	

Generalisation. For n-digit numbers with no leading 0 the same argument gives $9 \cdot 10^{n-2} \cdot 5 = 45 \cdot 10^{n-2}$: only the last digit is constrained, and it always has 5 options.

Free slots contribute 10 each; the final slot contributes 5.

Model solution to Problem 2(e)

Solution (what you write)	**Commentary (what it does)**
Statement. Count all strings of the form four letters + three-digit number, where each letter slot holds any of the 26 letters (letters may repeat) and the number does not start with 0.	*State the assumption that letters may repeat; the problem does not forbid it, and real course codes (e.g. ENGR, EEEE) do repeat letters.*
Solution. Each of the 4 letter slots has 26 options: 26^4 ways. The first digit has 9 options (1 to 9) and each of the other two digits has 10 options: $9 \cdot 10 \cdot 10$ ways. By the rule of product the count is $26^4 \cdot 9 \cdot 10^2$.	*This is exactly Figure 7.*
Answer. $\boxed{26^4 \cdot 9 \cdot 10^2 = 456\,976 \cdot 900 = 411\,278\,400}$.	

Part (f) needs inclusion-exclusion. The letter block has four slots, and "OK" can sit in positions 1-2, 2-3 or 3-4. Let A_1, A_2, A_3 be the sets of four-letter blocks with OK starting at position 1, 2, 3 respectively. We want the blocks in *none* of the A_i, so we compute $|A_1 \cup A_2 \cup A_3|$ and subtract from 26^4. Two of the sets can overlap only if their OK's do not collide.

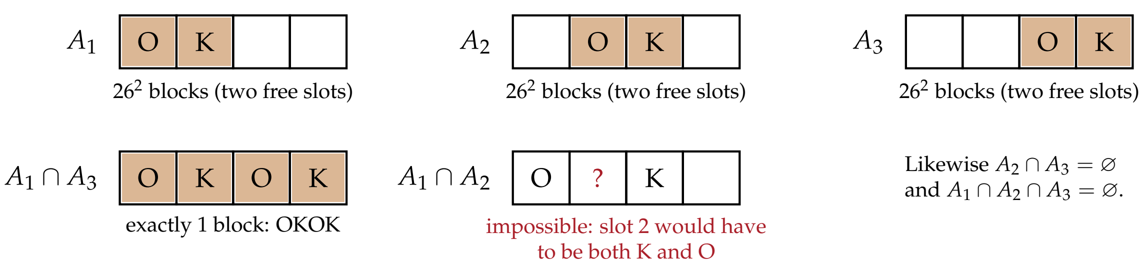


Figure 8. Problem 2(f): the three places where OK can occur, and which of them can occur together.

Model solution to Problem 2(f)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the strings of the form four letters + 339 (letters from the 26-letter alphabet, repetition allowed) whose four-letter block does not contain OK as a substring.	*The digit block is fixed, so only the letter block is counted.*

Solution. There are 26^4 four-letter blocks in total. We count the *bad* blocks, those containing OK, and subtract. For $i = 1, 2, 3$ let A_i be the set of blocks with O in slot i and K in slot $i + 1$. A block is bad exactly when it lies in $A_1 \cup A_2 \cup A_3$.

Single sets. In A_i two slots are fixed and the other two are free: $|A_i| = 26^2$ for each i.

Pairs. $A_1 \cap A_3$ consists of the single block OKOK, so $|A_1 \cap A_3| = 1$. $A_1 \cap A_2$ would need slot 2 to be both K and O, so $A_1 \cap A_2 = \varnothing$; likewise $A_2 \cap A_3 = \varnothing$.

Triple. $A_1 \cap A_2 \cap A_3 \subseteq A_1 \cap A_2 = \varnothing$.

By inclusion-exclusion, $|A_1 \cup A_2 \cup A_3| = 3 \cdot 26^2 - (1 + 0 + 0) + 0 = 3 \cdot 26^2 - 1$. Hence the number of good blocks is $26^4 - 3 \cdot 26^2 + 1$.

Define the sets, then compute singles, pairs and the triple separately. The only non-empty overlap is OKOK, and it must be added back because it was subtracted twice (once in $|A_1|$, once in $|A_3|$).

Answer. $\boxed{26^4 - 3 \cdot 26^2 + 1 = 456\,976 - 2028 + 1 = 454\,949}$.

Check. Shrink the alphabet to $\{O, K\}$. The formula becomes $2^4 - 3 \cdot 2^2 + 1 = 5$, and indeed the blocks over $\{O, K\}$ with no OK are exactly those of the form K. . . K O. . . O: KKKK, KKKO, KKOO, KOOO, OOOO. ✓

Shrinking the alphabet keeps the structure of the argument but makes the list small enough to write down.

2.4 Problem 3: combinatorial arguments for integrality

Problem 3

a) Provide a combinatorial argument to show that if n and k are positive integers with $n = 2k$, then $\frac{n!}{2^k}$ is an integer.
b) Provide a combinatorial argument to show that if n and k are positive integers with $n = 3k$, then $\frac{n!}{(3!)^k}$ is an integer.
c) Generalise the results from parts a) and b).

How to think about it. A "combinatorial argument" means: do not touch the algebra; instead find a set whose size is the given expression. The expression $\frac{n!}{2^k} = \frac{n!}{2!\,2! \cdots 2!}$ (k factors of 2!) has exactly the shape of the multiset formula from Section 1.3 with k types of objects, two of each type. So the set to consider is: *all arrangements of k pairs of identical symbols*. Figure 2 shows the case $k = 2$, $n = 4$: the $4!/(2!\,2!) = 6$ words made of A, A, B, B.

Model solution to Problem 3(a)

Solution (what you write)	Commentary (what it does)

Statement. If n and k are positive integers with $n = 2k$, then $\frac{n!}{2^k}$ is an integer.

Solution. Consider the $n = 2k$ symbols

$$x_1, x_1,\ x_2, x_2,\ \ldots,\ x_k, x_k,$$

that is, k different symbols, each appearing exactly twice. Let S be the set of all distinct arrangements (linear orderings) of these n symbols.
If all n symbols were distinguishable there would be $n!$ arrangements. Each of the k pairs consists of two indistinguishable copies, so every arrangement in S arises from exactly $2! \cdot 2! \cdots 2! = (2!)^k = 2^k$ of the $n!$ labelled arrangements (swap the two copies of x_1, or not; swap the two copies of x_2, or not; and so on). Therefore

$$|S| = \frac{n!}{\underbrace{2!\,2!\cdots 2!}_{k \text{ times}}} = \frac{n!}{2^k}.$$

Step (i): name the set. Step (ii): justify that its size is the expression, using the multiset formula, and explain the denominator in words. Notice $2! = 2$, which is why the denominator is 2^k.

Conclusion. $\dfrac{n!}{2^k}$ is the number of elements of the finite set S, hence it is a (positive) integer. ■

Step (iii): sizes of finite sets are integers.

Check. $k = 2$: $\dfrac{4!}{2^2} = 6$, and the six words AABB, ABAB, ABBA, BAAB, BABA, BBAA are exactly S (Figure 2). ✓

Model solution to Problem 3(b)

Solution (what you write)	**Commentary (what it does)**

Statement. If n and k are positive integers with $n = 3k$, then $\dfrac{n!}{(3!)^k}$ is an integer.

Solution. Consider the $n = 3k$ symbols

$$x_1, x_1, x_1,\ x_2, x_2, x_2,\ \ldots,\ x_k, x_k, x_k,$$

that is, k different symbols, each appearing exactly three times, and let S be the set of all distinct arrangements of these n symbols. By the formula for arrangements with repeated objects (k types, 3 identical objects of each type),

$$|S| = \frac{n!}{\underbrace{3!\,3!\cdots 3!}_{k \text{ times}}} = \frac{n!}{(3!)^k}.$$

Same set-up, three copies per symbol. Here $3! = 6$, so the claim is also that 6^k divides $(3k)!$.

Conclusion. $\dfrac{n!}{(3!)^k}$ counts the elements of the finite set S, hence it is an integer. ■

Check. $k = 2$: $\dfrac{6!}{6^2} = \dfrac{720}{36} = 20$; the words with three A's and three B's are $\binom{6}{3} = 20$ (choose the positions of the A's). ✓

The two counts agree, as they must.

Model solution to Problem 3(c)

Solution (what you write) | **Commentary (what it does)**

Statement (generalisation). If n, m and k are positive integers with $n = m \cdot k$, then $\dfrac{n!}{(m!)^k}$ is an integer.

Parts (a) and (b) are the cases $m = 2$ and $m = 3$.

Solution. Consider k different symbols $x_1, \ldots, x_k$, each appearing exactly m times, for a total of $n = mk$ symbols, and let S be the set of all distinct arrangements of them. By the formula for arrangements with repeated objects,

The proof is word for word the same; only the multiplicity changes.

$$|S| = \frac{n!}{\underbrace{m!\, m! \cdots m!}_{k \text{ times}}} = \frac{n!}{(m!)^k}.$$

Since $|S|$ is the size of a finite set, $\dfrac{n!}{(m!)^k}$ is an integer. ∎

Further generalisation. The groups need not have equal size: for any positive integers $n_1, \ldots, n_r$ with $n_1 + \cdots + n_r = n$, the number $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$ is an integer, because it counts the arrangements of n_1 copies of x_1, n_2 copies of x_2, ..., n_r copies of x_r. Part (c) is the special case $n_1 = \cdots = n_r = m$ with $r = k$.

This is the multinomial coefficient. As a corollary, $(m!)^k$ divides $(mk)!$ for all positive integers m, k.

Why not just do the algebra?

One *could* prove $2^k \mid (2k)!$ by noticing that $(2k)! = 2 \cdot 4 \cdot 6 \cdots (2k) \cdot (\text{odd numbers})$ and $2 \cdot 4 \cdots (2k) = 2^k k!$. That works for $m = 2$ but becomes messy for general m. The counting argument costs one sentence, works for every m, and explains *what the integer is*: it is a number of arrangements. This is what the question means by "provide a combinatorial argument".

2.5 Problem 4: binary and ternary strings

Problem 4

a) How many bytes (binary strings of length 8) contain exactly two 1's?
b) How many ternary strings (strings of 0's, 1's and 2's) of length 8 contain exactly two 1's?
c) How many binary strings of length n contain exactly four 1's? (Assume $n > 3$.)
d) How many ternary strings of length n contain exactly four 1's? (Assume $n > 3$.)

How to think about it. A string is determined by *where* its 1's are and *what* sits in the other positions. "Exactly two 1's" means: choose the two positions for the 1's (a combination, Figure 3), and then fill every remaining position with a symbol that is *not* 1. For binary strings the remaining symbol is forced (0); for ternary strings each remaining position has two options (0 or 2).

$$\underbrace{\boxed{0/2}\,\boxed{0/2}\,\boxed{1}\,\boxed{0/2}\,\boxed{0/2}\,\boxed{1}\,\boxed{0/2}\,\boxed{0/2}}_{\text{choose 2 of the 8 positions for the 1's, then fill each other position from } \{0,2\}} \qquad \binom{8}{2} \cdot 2^6$$

Figure 9. Problem 4(b): positions of the 1*'s first, then the six remaining positions filled from* $\{0,2\}$.

Model solution to Problem 4(a)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the binary strings of length 8 with exactly two 1's.	
Solution. A binary string of length 8 with exactly two 1's is completely determined by the set of two positions (out of 8) that hold the 1's; all other positions must hold 0. Different position sets give different strings, and every such string arises this way. Hence the number of strings equals the number of 2-element subsets of $\{1,\ldots,8\}$, which is $\binom{8}{2}$.	*Explain the correspondence in both directions ("determined by" and "every such string arises"); that is what makes the count exact.*
Answer. $\boxed{\binom{8}{2} = \frac{8\cdot 7}{2} = 28}$.	

Model solution to Problem 4(b)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the ternary strings of length 8 with exactly two 1's.	
Solution. First choose the two positions that hold the 1's: $\binom{8}{2}$ ways. The remaining 6 positions must *not* hold a 1 (otherwise the string would have more than two 1's), so each of them holds 0 or 2: 2 options each, 2^6 ways in total. By the rule of product the count is $\binom{8}{2}\cdot 2^6$.	*The word "exactly" is what forces the other positions to avoid 1. Writing* 3^6 *here would count strings with* at least *two 1's, and would overcount them.*
Answer. $\boxed{\binom{8}{2}\cdot 2^6 = 28\cdot 64 = 1792}$.	
Check. Length 3 instead of 8: $\binom{3}{2}\cdot 2^1 = 6$ strings, namely 110, 112, 101, 121, 011, 211. ✓	

Model solution to Problem 4(c)

Solution (what you write)	**Commentary (what it does)**
Statement. Count the binary strings of length n ($n > 3$) with exactly four 1's.	
Solution. Such a string is determined by the 4-element set of positions holding the 1's (all other positions hold 0). Hence the count is the number of 4-element subsets of $\{1,\ldots,n\}$, namely $\binom{n}{4}$. The assumption $n > 3$ guarantees that at least one such subset exists.	*Same argument as (a) with 8 replaced by* n *and 2 by 4.*
Answer. $\boxed{\binom{n}{4}}$.	

Model solution to Problem 4(d)

Solution (what you write)	Commentary (what it does)
Statement. Count the ternary strings of length n ($n > 3$) with exactly four 1's.	
Solution. Choose the 4 positions for the 1's: $\binom{n}{4}$ ways. Each of the remaining $n-4$ positions holds 0 or 2: 2^{n-4} ways. By the rule of product the count is $\binom{n}{4} \cdot 2^{n-4}$.	
Answer. $\boxed{\binom{n}{4} \cdot 2^{n-4}}$.	
Check. The same reasoning with two 1's instead of four gives $\binom{n}{2}\, 2^{n-2}$, which at $n = 8$ is $28 \cdot 64 = 1792$, matching (b). ✓	*The general pattern:* $\binom{n}{r} \cdot 2^{n-r}$ *ternary strings have exactly r ones.*

The general principle behind Problem 4

Over an alphabet of size q, the number of strings of length n with exactly r occurrences of one particular symbol is

$$\binom{n}{r}\,(q-1)^{n-r}\,:$$

choose where the symbol goes, then fill the other positions from the $q-1$ remaining symbols. Binary strings are $q = 2$ (so $(q-1)^{n-r} = 1$), ternary strings are $q = 3$. This is also the reason the binomial theorem has the shape $(x+y)^n = \sum_r \binom{n}{r} x^{n-r} y^r$: a term $x^{n-r}y^r$ arises once for each choice of which r of the n factors contribute a y.

3 Mistakes, checklist and exercises

3.1 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **Confusing arrangements with selections.** Ask: does swapping two chosen objects give a new outcome? If yes, use $P(n,r)$ or $n!$; if no, use $\binom{n}{r}$. A committee is a selection; a ranking is an arrangement.
2. **Forgetting the leading-digit restriction, or applying it to the wrong slot.** "Cannot start with 0" affects only the first digit (9 options, or 7 in Problem 2(c)); the other slots keep 0 as an option.
3. **"Exactly" treated as "at least".** With exactly two 1's, the other positions must avoid 1: $\binom{8}{2}\, 2^6$, not $\binom{8}{2}\, 3^6$ (Problem 4(b)).
4. **Gluing to count adjacency when there are three or more special objects.** Treating two e's as one block counts arrangements with three adjacent e's several times. Use the gap method instead (Problem 1).
5. **Forgetting the add-back in inclusion-exclusion.** $26^4 - 3 \cdot 26^2$ subtracts OKOK twice; the $+1$ is not optional (Problem 2(f)).
6. **Multiplying when the number of options depends on earlier choices.** Either fill the

restricted slot first (Section 1.1) or split into cases and add. In Problem 2(d) the rule of product is fine only because the number of options for the last digit is 5 in every case.

7. **Dividing by $k!$ for objects that are distinguishable, or forgetting to divide for identical ones.** Problem 1(a) vs 1(b): identical e's give $\binom{5}{5}$, labelled e's give $5!$.
8. **A number with no argument.** "$4! = 24$" is not a solution. Say what is being counted and which rule gives each factor.

Spot the error 1: Problem 1(b) by the complement

"Solution." There are 9 distinct symbols, so $9!$ arrangements in total. Arrangements with two adjacent e's: glue two e's into a block; then there are 8 objects to arrange, so $8!$ arrangements, and the block can be chosen in $\binom{5}{2}$ ways and ordered internally in 2 ways. Answer: $9! - \binom{5}{2} \cdot 2 \cdot 8!$.
Where is the mistake? (Answer in Appendix A.) Hint: compute the "answer" and compare with 2880.

Spot the error 2: Problem 2(c)

"Solution." Eight digits are allowed $(0, 1, 2, 4, 5, 6, 7, 8)$ and there are three slots, so the answer is $8^3 = 512$.
Where is the mistake? (Answer in Appendix A.)

Spot the error 3: Problem 4(b)

"Solution." Choose the two positions for the 1's: $\binom{8}{2} = 28$. Fill the other six positions with any of $0, 1, 2$: $3^6 = 729$. Answer: $28 \cdot 729 = 20\,412$.
Where is the mistake? (Answer in Appendix A.) Hint: there are only $3^8 = 6561$ ternary strings of length 8.

3.2 Checklist before you hand in a counting solution

Checklist

- ☐ I stated precisely **what is being counted** and what makes an outcome valid.
- ☐ I decided explicitly: **order** matters or not, **repetition** allowed or not, which objects are **identical**.
- ☐ Every factor in my product is justified by a named rule, and the number of options at each step **does not depend on earlier choices** (or I split into cases and added).
- ☐ For "at least", "must contain" or "does not contain" I used the **complement**, and for two or more unwanted properties I applied **inclusion-exclusion** with all overlap terms.
- ☐ For "no two adjacent" I placed the other objects first and counted **gaps**.
- ☐ For a combinatorial argument I **named the set**, justified its size, and concluded that the expression is an integer.
- ☐ The **answer is boxed**, as an expression and as a number where reasonable, and I **checked** it on a tiny case or by a second method.

3.3 Exercises

Write each solution in full, using the template.

E1. How many arrangements of the letters of BANANAS have no two A's adjacent? (This was done in the lecture; write it up in the standard format.)

E2. For the letters of MISSISSIPPI, count (a) all arrangements, (b) the arrangements with no two I's adjacent, (c) the arrangements in which the four S's are consecutive.

E3. Consider all 4-digit PINs 0000 to 9999. How many contain (a) at least one 7? (b) at least one 7 and at least one 0?

E4. How many binary strings of length 10 contain exactly three 1's, no two of which are adjacent? Then find a formula for length n and k ones.

E5. How many 5-letter words over the 26-letter alphabet (letters may repeat) do not contain the substring AB?

E6. Use Problem 3(c) to show that $(n!)!$ is divisible by $(n!)^{(n-1)!}$ for every positive integer n.

E7. How many three-digit numbers (no leading 0) have a digit sum divisible by 5?

E8. How many ternary strings of length n ($n \geq 6$) contain exactly four 1's and exactly two 2's?

E9. The symbols a, b, c, d and three e's are to be arranged with no two e's adjacent. Count the arrangements (a) if the e's are identical, (b) if they are labelled e_1, e_2, e_3.

E10. How many ternary strings of length 8 contain (a) at least one 1? (b) at least one 1 and at least one 2?

A Answers to the "spot the error" puzzles

Spot the error 1 (gluing). The "answer" is $9! - 10 \cdot 2 \cdot 8! = 362\,880 - 806\,400 < 0$, which is impossible, so something is badly wrong. The gluing count $\binom{5}{2} \cdot 2 \cdot 8!$ is not the number of arrangements with *some* adjacent pair; it counts each arrangement once *for every adjacent pair it contains*. An arrangement such as $e_1e_2e_3\,a\,\ldots$ contains two adjacent pairs (e_1e_2 and e_2e_3) and is counted twice; $e_1e_2e_3e_4e_5\,abcd$ is counted four times. Fixing this by inclusion-exclusion over all pairs is possible but painful. The gap method avoids the issue entirely, which is why it is the right tool whenever three or more objects must be kept apart. (Gluing does work for the complement when there are exactly *two* special objects, since then there is only one possible adjacent pair.)

Spot the error 2 (leading zero). The first digit cannot be 0, so it has only 7 options. Correct count: $7 \cdot 8 \cdot 8 = 448$, not 512. The 64 missing numbers are exactly those starting with 0 (0 followed by two allowed digits: $8 \cdot 8 = 64$), and $512 - 64 = 448$.

Spot the error 3 (exactly vs at least). Filling the other six positions from $\{0, 1, 2\}$ allows more 1's, so the strings counted do not have *exactly* two 1's. Worse, a string with three 1's is counted three times (once for each choice of which two of its 1's were "the chosen ones"), so the number is not even the number of strings with at least two 1's. The total 20 412 exceeds the number of all ternary strings of length 8, which confirms the overcount. Correct count: $\binom{8}{2} \cdot 2^6 = 1792$.