Mathematical Induction and Strong Induction

How the technique works, and how to write a proof that is complete and correct

Lecture notes (by Parsa Kamalipour)

What you will be able to do after this

1. Explain *why* a proof by induction proves infinitely many statements at once.
2. Write a proof by induction in the standard format: **Statement** $\rightarrow$ **Solution** (set-up) $\rightarrow$ **Base case** $\rightarrow$ **Inductive hypothesis** $\rightarrow$ **Inductive step** $\rightarrow$ **Conclusion**, using the proper proof vocabulary.
3. Recognise when ordinary induction is not enough, and use *strong* induction instead.
4. Decide how many base cases a strong-induction proof needs, and avoid the classic mistakes.

Contents

1 Why do we need induction?

Look at the following sums:

$$1 = 1, \qquad 1+2 = 3, \qquad 1+2+3 = 6, \qquad 1+2+3+4 = 10, \qquad 1+2+3+4+5 = 15.$$

Compare them with the numbers $\dfrac{n(n+1)}{2}$ for $n = 1, 2, 3, 4, 5$:

$$\frac{1\cdot 2}{2} = 1, \qquad \frac{2\cdot 3}{2} = 3, \qquad \frac{3\cdot 4}{2} = 6, \qquad \frac{4\cdot 5}{2} = 10, \qquad \frac{5\cdot 6}{2} = 15.$$

They agree, so we *suspect* that

$$1 + 2 + \cdots + n \;=\; \frac{n(n+1)}{2} \qquad \text{for every integer } n \geq 1.$$

But five checks (or five million checks) do not prove a statement about *every* n. The claim "for every integer $n \geq 1$" is really an **infinite list of statements**: one statement for $n = 1$, one for $n = 2$, one for $n = 3$, and so on forever. We cannot check them one at a time; we need a single, finite argument that covers all of them.

Mathematical induction is exactly such an argument. It is the standard tool for proving statements of the form "for every integer $n \geq n_0$, some property $P(n)$ holds", and it is the technique behind almost every proof about recursion, algorithms, sums, and data structures.

2 Mathematical induction

2.1 The idea: falling dominoes

The intuition

Imagine an infinite row of dominoes, numbered $1, 2, 3, \ldots$, standing upright. Domino n "falls" when the statement $P(n)$ is proved. To make *every* domino fall you need exactly two things:

(i) you push the **first domino** over, and

(ii) the dominoes are placed so that **whenever any domino falls, it knocks over the next one**.

Then domino 1 falls (you pushed it), so domino 2 falls, so domino 3 falls, … and no domino stays standing.

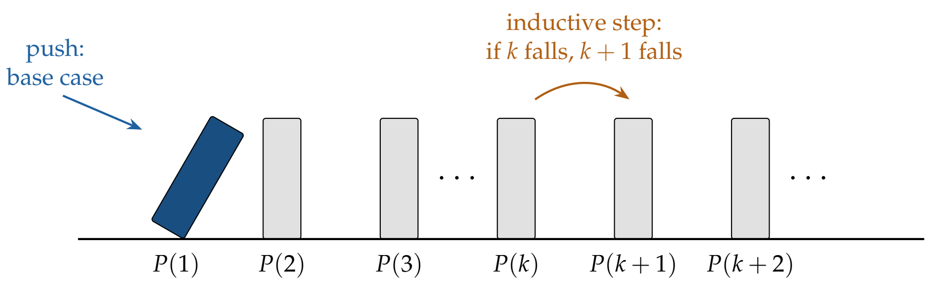

Figure 1. A proof by induction: one push (the base case) plus one rule that propagates from each domino to the next (the inductive step).

In a proof, the two ingredients are called:

- the **base case**: prove $P(n_0)$ directly for the first value n_0 (usually $n_0 = 0$ or 1 but not always);
- the **inductive step**: prove that for every integer $k \geq n_0$, *if* $P(k)$ is true *then* $P(k+1)$ is true. The assumption "$P(k)$ is true" is called the **inductive hypothesis**.

Notice that in the inductive step we are *not* assuming what we want to prove. We are proving an *implication*: "*if* $P(k)$ holds, *then* $P(k+1)$ holds". Proving this implication for every k does not, by itself, tell us that any particular $P(k)$ is true; it only tells us how truth propagates from one case to the next. The base case is what starts the chain.

2.2 A simple example

Example 1 (the simple example): the sum $1+2+\cdots+n$

Statement. For every integer $n \geq 1$, $\quad 1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$.

First, informally. Let $P(n)$ be the equation $1+2+\cdots+n = \frac{n(n+1)}{2}$.

The first domino. $P(1)$ says $1 = \frac{1\cdot 2}{2}$, which is true.

One domino knocks over the next. Suppose we already know that $P(k)$ is true for some particular k, i.e. $1+2+\cdots+k = \frac{k(k+1)}{2}$. What does that tell us about $P(k+1)$? The left side of $P(k+1)$ is the left side of $P(k)$ plus one more term, $k+1$. So

$$1+2+\cdots+k+(k+1) \;=\; \underbrace{\frac{k(k+1)}{2}}_{\text{from } P(k)} + (k+1) \;=\; (k+1)\left(\frac{k}{2}+1\right) \;=\; \frac{(k+1)(k+2)}{2},$$

and $\frac{(k+1)(k+2)}{2}$ is exactly the right side of $P(k+1)$. So $P(k)$ really does force $P(k+1)$.

That is the whole argument. Now let us write it the way it should be written in an assignment or exam. The proof on the left is the finished product; the commentary on the right explains the job of each sentence.

Model proof of Example 1 (written in the standard format)

Proof (what you write)	Commentary (what it does)
Statement. For every integer $n \geq 1$, $1+2+\cdots+n = \dfrac{n(n+1)}{2}$.	*State exactly what is being proved, including the range of n.*
Solution. Let $P(n)$ be the statement $$1+2+\cdots+n = \frac{n(n+1)}{2}.$$ We prove that $P(n)$ holds for every integer $n \geq 1$ by induction on n.	*Give the statement a name, $P(n)$, and announce the method ("by induction on n") and the starting value.*
Base case ($n = 1$). The left-hand side of $P(1)$ is 1. The right-hand side is $\frac{1\cdot(1+1)}{2} = 1$. The two sides are equal, so $P(1)$ holds.	*Prove the first case directly. Compute* both *sides separately and then compare; do not just write "$1 = 1$".*

Inductive hypothesis. Let $k \geq 1$ be an arbitrary integer and assume that $P(k)$ holds, that is,

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Fix one arbitrary k in the range and assume $P(k)$ for that k only. Write out what $P(k)$ says.

Inductive step. We must show that $P(k+1)$ holds, that is,

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Starting from the left-hand side,

$$\begin{aligned} &1 + 2 + \cdots + k + (k+1) \\ &= \big(1 + 2 + \cdots + k\big) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{[by the inductive hypothesis]} \\ &= (k+1)\Big(\frac{k}{2} + 1\Big) = \frac{(k+1)(k+2)}{2}, \end{aligned}$$

which is precisely the right-hand side of $P(k+1)$. Hence $P(k+1)$ holds.

Write out the target $P(k+1)$ before you start. Then transform one side into the other; the moment you use the assumption, say so. End by pointing out that you have reached the target.

Conclusion. Since the base case holds and $P(k)$ implies $P(k+1)$ for every integer $k \geq 1$, by the principle of mathematical induction $P(n)$ holds for every integer $n \geq 1$. ■

Close the argument by naming the principle you used. The square marks the end of the proof.

2.3 The actual explanation: the principle and how to write it

Principle of Mathematical Induction

Let $P(n)$ be a statement about the integer n, and let n_0 be an integer. Suppose that

(i) **(base case)** $P(n_0)$ is true, and

(ii) **(inductive step)** for every integer $k \geq n_0$: if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for every integer $n \geq n_0$.

Why does this prove everything? The base case gives $P(n_0)$. The inductive step with $k = n_0$ says "$P(n_0)$ implies $P(n_0+1)$", so $P(n_0+1)$ is true. The inductive step with $k = n_0 + 1$ then gives $P(n_0+2)$, and so on. Every integer $n \geq n_0$ is reached after finitely many steps, so no case is missed.

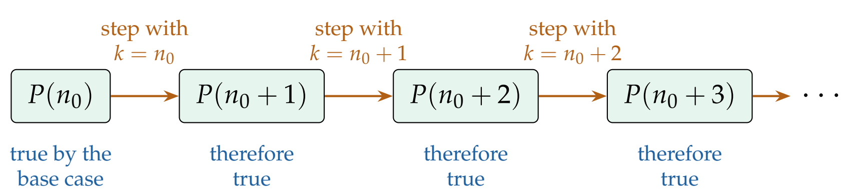

Figure 2. The chain of implications produced by the inductive step, started by the base case.

Remark: where the principle comes from (optional)

Induction is one of the basic axioms describing the natural numbers. It is equivalent to the *well-ordering principle*: every non-empty set of non-negative integers has a smallest element. Here is the connection. Suppose (i) and (ii) hold but $P(n)$ fails for some $n \geq n_0$. Let m be the *smallest* such n. By (i), $m \neq n_0$, so $m - 1 \geq n_0$, and $P(m-1)$ is true because m was the smallest failure. But then (ii) with $k = m - 1$ gives $P(m)$, a contradiction. So $P(n)$ never fails.

The base case does not have to be 0 or 1. The principle works from any starting value n_0. If a statement is only true from $n_0 = 5$ onward (we will see one in Example 2), then the base case is $n = 5$, the inductive hypothesis is made for an arbitrary $k \geq 5$, and the conclusion is "for every $n \geq 5$". Always match these three places.

How to write the proof. Every induction proof has the same skeleton. Learn it, and use the same headings every time; graders (and readers) will thank you.

Writing template for a proof by induction

Statement. For every integer $n \geq n_0$, *[the claim about n]*.
Solution. Let $P(n)$ be the statement "*[the claim, written with n]*". We prove that $P(n)$ holds for every integer $n \geq n_0$ by induction on n.
Base case ($n = n_0$). We must show $P(n_0)$. *[Verify it directly. For an equation, compute both sides. For an inequality or a divisibility claim, show the numbers.]* Hence $P(n_0)$ holds.
Inductive hypothesis. Let $k \geq n_0$ be an arbitrary integer and assume that $P(k)$ holds, that is, *[write out $P(k)$ explicitly]*.
Inductive step. We must show that $P(k+1)$ holds, that is, *[write out $P(k+1)$ explicitly]*. *[Start from one side of $P(k+1)$, or from the quantity in $P(k+1)$, and work until you reach the other side. Somewhere in the middle you will use $P(k)$; write "by the inductive hypothesis" at that exact spot.]* This is precisely $P(k+1)$, so $P(k+1)$ holds.
Conclusion. Since $P(n_0)$ holds and $P(k)$ implies $P(k+1)$ for every integer $k \geq n_0$, by the principle of mathematical induction $P(n)$ holds for every integer $n \geq n_0$. ■

Proof phrasebook: words and sentences that belong in an induction proof

Where	What to write
Setting up	"Let $P(n)$ be the statement that ..." "We proceed by induction on n." "We show that $P(n)$ holds for all integers $n \geq n_0$."
Base case	"Base case ($n = n_0$). We must show ..." "The left-hand side equals ..., and the right-hand side equals ..., so they are equal." "Hence $P(n_0)$ holds."
Hypothesis	"Let $k \geq n_0$ be an arbitrary integer." "Assume that $P(k)$ holds, that is, ..." "Suppose (for the purpose of induction) that ..."
Step: the goal	"We must show that $P(k+1)$ holds, that is, ..." "Our goal is to prove ..."
Step: using $P(k)$	"By the inductive hypothesis, ..." "Substituting the inductive hypothesis, ..." "Applying $P(k)$ to the bracketed sum, ..."
Step: algebra	"Factoring, ..." "Rearranging, ..." "Since $k \geq 5$, we have ..." "Because q is an integer, so is $4q + 1$."
Step: finishing	"...which is precisely the statement $P(k+1)$." "This completes the inductive step."
Concluding	"By the principle of mathematical induction, $P(n)$ holds for all integers $n \geq n_0$." "This completes the proof." ■
Logic words	*therefore, hence, thus, so, it follows that, consequently, this shows that, in particular, since, because.*

2.4 Comprehensive examples

Sums are the friendliest use of induction. Two other very common uses are *inequalities* and *divisibility*. Each has its own small trick, shown below. In both examples we first do some *scratch work* (thinking that does not appear in the final proof) and then write the clean proof.

Example 2 (comprehensive): an inequality with a base case other than 1

Statement. For every integer $n \geq 5$, $\quad 2^n > n^2$.

Scratch work (not part of the proof). First check where the claim starts:

n	1	2	3	4	5	6	7
2^n	2	4	8	16	32	64	128
n^2	1	4	9	16	25	36	49
$2^n > n^2$?	✓	✗	✗	✗	✓	✓	✓

The claim is false for $n = 2, 3, 4$, so the statement correctly starts at $n = 5$, and our base case must be $n = 5$ (not $n = 1$).

Next, plan the inductive step. We will know $2^k > k^2$ and want $2^{k+1} > (k+1)^2$. The natural move is $2^{k+1} = 2 \cdot 2^k > 2k^2$ (using the hypothesis). So we would be done if $2k^2 \geq (k+1)^2$, i.e. if $k^2 \geq 2k + 1$. Is that true for $k \geq 5$? Yes: $k^2 = k \cdot k \geq 5k = 2k + 3k \geq 2k + 15 > 2k + 1$. That is the extra fact we need to prove inside the step; now we write it all up.

Model proof of Example 2

Proof (what you write)	Commentary (what it does)
Statement. For every integer $n \geq 5$, $2^n > n^2$.	
Solution. Let $P(n)$ be the statement "$2^n > n^2$". We prove that $P(n)$ holds for every integer $n \geq 5$ by induction on n.	*The starting value 5 appears here, in the base case, in the hypothesis and in the conclusion.*
Base case ($n = 5$). We have $2^5 = 32$ and $5^2 = 25$. Since $32 > 25$, $P(5)$ holds.	*Compute both sides and compare.*
Inductive hypothesis. Let $k \geq 5$ be an arbitrary integer and assume that $P(k)$ holds, that is, $2^k > k^2$.	
Inductive step. We must show that $P(k+1)$ holds, that is, $2^{k+1} > (k+1)^2$. First observe that, because $k \geq 5$,	*For inequalities you build a* chain*: $2^{k+1} > \cdots > \cdots = (k+1)^2$. Every link must point the same way ($>$ or $\geq$), and at least one link must be strict to get "$>$" at the end. The side fact $(*)$ is proved right where it is needed, and it is where the assumption $k \geq 5$ is used.*

$$k^2 = k \cdot k \;\geq\; 5k \;=\; 2k + 3k \;\geq\; 2k + 15 \;>\; 2k + 1. \qquad (*)$$

Therefore

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^2 && \text{[by the inductive hypothesis]} \\ &= k^2 + k^2 \\ &> k^2 + 2k + 1 && \text{[by } (*)\text{]} \\ &= (k+1)^2. \end{aligned}$$

Hence $2^{k+1} > (k+1)^2$, which is $P(k+1)$.

Conclusion. By the principle of mathematical induction, $2^n > n^2$ for every integer $n \geq 5$. ∎

Example 3 (comprehensive): a divisibility statement

Statement. For every integer $n \geq 0$, the number $4^n - 1$ is divisible by 3.

Scratch work. Recall the definition: "3 divides m" (written $3 \mid m$) means $m = 3q$ for some integer q. Check a few values: $4^0 - 1 = 0 = 3 \cdot 0$, $4^1 - 1 = 3 = 3 \cdot 1$, $4^2 - 1 = 15 = 3 \cdot 5$, $4^3 - 1 = 63 = 3 \cdot 21$. Good.

For the step we will know $4^k - 1 = 3q$ and we must show $3 \mid 4^{k+1} - 1$. The trick for divisibility proofs is to *force the expression from the hypothesis to appear*: $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4 \cdot 4^k - 4 + 3 = 4(4^k - 1) + 3$. Now $4^k - 1$ can be replaced by $3q$.

Model proof of Example 3

Proof (what you write)	Commentary (what it does)

Statement. For every integer $n \geq 0$, $3 \mid (4^n - 1)$.

Solution. Let $P(n)$ be the statement "$4^n - 1$ is divisible by 3", i.e. "$4^n - 1 = 3q$ for some integer q". We prove that $P(n)$ holds for every integer $n \geq 0$ by induction on n.

Unfold the definition of "divisible" inside $P(n)$; you will need the concrete form $3q$ in the step.

Base case ($n = 0$). $4^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$, and 0 is an integer. Hence $3 \mid (4^0 - 1)$ and $P(0)$ holds.

A base case of 0 is perfectly fine; just make sure the statement is true there.

Inductive hypothesis. Let $k \geq 0$ be an arbitrary integer and assume that $P(k)$ holds, that is, $4^k - 1 = 3q$ for some integer q.

Name the integer (q) so you can use it in the step.

Inductive step. We must show that $P(k+1)$ holds, that is, that $4^{k+1} - 1$ is a multiple of 3. We compute

$$\begin{aligned}
4^{k+1} - 1 &= 4 \cdot 4^k - 1 \\
&= 4 \cdot 4^k - 4 + 3 \\
&= 4\,(4^k - 1) + 3 \\
&= 4 \cdot 3q + 3 && \text{[by the inductive hypothesis]} \\
&= 3\,(4q + 1).
\end{aligned}$$

"Subtract 4 and add 3" is the move that makes $4^k - 1$ appear. Finish by showing the result is $3 \times$ (an integer), and say why it is an integer.

Since q is an integer, $4q + 1$ is an integer, so $4^{k+1} - 1$ is divisible by 3. Hence $P(k+1)$ holds.

Conclusion. By the principle of mathematical induction, $4^n - 1$ is divisible by 3 for every integer $n \geq 0$. ■

Example 4 (optional reading): induction is not only algebra

Statement. For every integer $n \geq 1$, a $2^n \times 2^n$ board with *any one* square removed can be completely covered by L-shaped trominoes (three squares in an L shape), with no overlaps.

Solution. Let $P(n)$ be the statement "every $2^n \times 2^n$ board with any one square removed can be tiled by L-trominoes". We use induction on $n \geq 1$.
Base case ($n = 1$). A 2×2 board with one square removed is itself an L-tromino, so $P(1)$ holds.
Inductive hypothesis. Let $k \geq 1$ and assume $P(k)$: every $2^k \times 2^k$ board with any one square removed can be tiled.
Inductive step. Take a $2^{k+1} \times 2^{k+1}$ board with one square removed. Cut it into four $2^k \times 2^k$ quadrants. The removed square lies in one quadrant. Place one tromino at the centre of the board so that it covers exactly one square in each of the *other three* quadrants (Figure 3). Now every quadrant is a $2^k \times 2^k$ board with exactly one square missing, so by the inductive hypothesis each quadrant can be tiled. Together with the central tromino this tiles the whole board, proving $P(k+1)$.
Conclusion. By induction, $P(n)$ holds for all $n \geq 1$. ■
Note. The hypothesis is used four times (once per quadrant), and it matters that $P(k)$ says "*any* square removed": that is what lets us apply it to quadrants whose missing square is in the middle.

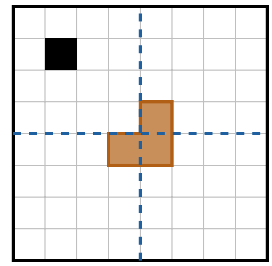

Figure 3. The inductive step for $k = 2$: after placing the central tromino, each quadrant is a 4×4 board missing one square.

2.5 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **No base case, or the wrong base case.** The chain needs a first domino. Also, the base case must be the first value in the statement ($n = 5$ in Example 2, not $n = 1$).
2. **Assuming the conclusion.** Writing "Assume $P(n)$ holds for all n" in the hypothesis is circular. The hypothesis is for *one arbitrary* k: "Let $k \geq n_0$ and assume $P(k)$."
3. **Not writing out $P(k+1)$.** If you do not state the target, you cannot check that you reached it. Substitute $k+1$ carefully: $\frac{(k+1)(k+2)}{2}$, not $\frac{(k+1)k}{2}$.
4. **Working on both sides of the goal as if it were true.** Do not start from the equation $P(k+1)$ and manipulate it until you get "$0 = 0$". Start from one side (or from the quantity) and transform it into the other.
5. **Using the hypothesis without saying so.** The line where $P(k)$ enters is the heart of the proof; label it.
6. **An inductive step that silently fails for a small k.** See the "horses" puzzle below: the argument must work for *every* $k \geq n_0$, including the very first one.

Spot the error: "all horses have the same colour"

Claim. In any set of $n \geq 1$ horses, all horses have the same colour. *"Proof."* Base case $n = 1$: one horse has one colour. Inductive step: assume any k horses share a colour. Take $k+1$ horses $h_1, \ldots, h_{k+1}$. The horses $h_1, \ldots, h_k$ share a colour by the hypothesis, and so do $h_2, \ldots, h_{k+1}$. Both groups contain h_2, so all $k+1$ horses share the colour of h_2. ■

Where is the mistake? (Answer in Appendix A.) Hint: try the inductive step with $k = 1$.

2.6 Quick recap: ordinary induction

Ordinary induction in one breath

- Use it for statements of the form "for every integer $n \geq n_0$, $P(n)$".
- **Base case:** prove $P(n_0)$ directly.
- **Inductive hypothesis:** let $k \geq n_0$ be arbitrary; assume $P(k)$ (write it out).
- **Inductive step:** write out $P(k+1)$; transform it using $P(k)$ (say "by the inductive hypothesis").
- **Conclusion:** "by the principle of mathematical induction, $P(n)$ holds for all $n \geq n_0$."
- Tricks: sums, peel off the last term; inequalities, build a one-directional chain; divisibility, force the hypothesis expression to appear and end with $d \times$ (integer).

3 Strong induction

3.1 The idea: when the previous domino is not enough

In ordinary induction, the inductive step gets to use only the *immediately previous* case $P(k)$. Sometimes that is not the case you need. Consider this claim:

Every integer $n \geq 2$ can be written as a product of one or more prime numbers.

(For example $60 = 2 \cdot 2 \cdot 3 \cdot 5$, and a prime such as 7 is a "product" of just one prime.) Try the inductive step in the ordinary way. Assume $P(k)$: "k is a product of primes". Now look at $k+1$. If $k+1$ is prime we are done. If $k+1$ is composite, then $k+1 = a \cdot b$ for some integers a, b with $2 \leq a, b \leq k$. To factor $k+1$ we would like to factor a and b. But our hypothesis only talks about k, and a and b are (usually) smaller than k. We are stuck: knowing that 59 is prime tells us nothing about how to factor 60.

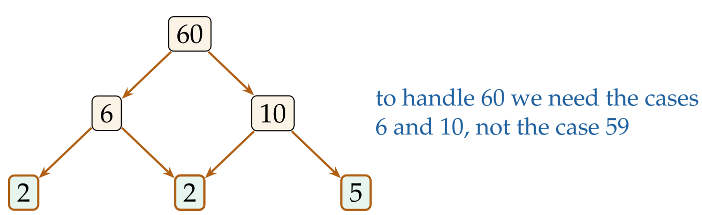

Figure 4. Factoring 60 recursively: the cases we need are much smaller than $59 = 60 - 1$.

What would fix the argument? If the hypothesis said "*every* integer from 2 up to k is a product of primes", then a and b would both be covered, and the step would go through. That is exactly what strong induction allows.

The intuition

Same infinite row of dominoes, same goal. The only change is the rule for the inductive step: to show that domino $k+1$ falls, you may use the fact that **all** dominoes $n_0, n_0+1, \ldots, k$ have already fallen, not just domino k. Nothing is lost: the dominoes still fall in order, one at a time, starting from the base case(s). You simply have more information available at each step.

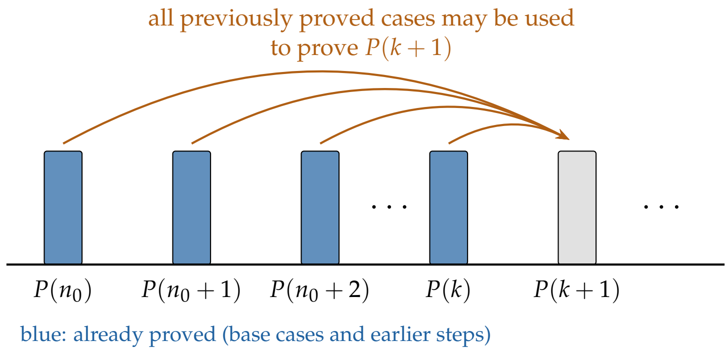

Figure 5. Strong induction: the step for $P(k+1)$ may rely on any (or all) of $P(n_0), \ldots, P(k)$.

3.2 A simple example

Example 5 (the simple example): prime factorisation exists

Statement. Every integer $n \geq 2$ can be written as a product of one or more primes.

Model proof of Example 5 (strong induction, standard format)

Proof (what you write)	**Commentary (what it does)**
Statement. Every integer $n \geq 2$ can be written as a product of one or more primes.	
Solution. Let $P(n)$ be the statement "n can be written as a product of one or more primes". We prove that $P(n)$ holds for every integer $n \geq 2$ by *strong* induction on n.	*Say "strong induction" so the reader expects the stronger hypothesis.*
Base case ($n = 2$). The number 2 is prime, so it is a product of one prime. Hence $P(2)$ holds.	*One base case is enough here: the step below works for every $k \geq 2$.*
Inductive hypothesis. Let $k \geq 2$ be an arbitrary integer and assume that $P(j)$ holds for *every* integer j with $2 \leq j \leq k$; that is, every integer from 2 to k is a product of primes.	*This is the strong hypothesis: all cases from the base case up to k, not just $P(k)$.*

Inductive step. We must show that $P(k+1)$ holds. There are two cases.
Case 1: $k+1$ is prime. Then $k+1$ is a product of one prime, so $P(k+1)$ holds.
Case 2: $k+1$ is composite. Then $k+1 = a \cdot b$ for some integers a, b with $2 \le a \le b$. Since $b \ge 2$, we have $a = (k+1)/b \le (k+1)/2 < k+1$, so $a \le k$; similarly $b \le k$. Thus $2 \le a \le k$ and $2 \le b \le k$, so by the inductive hypothesis $a = p_1 p_2 \cdots p_r$ and $b = q_1 q_2 \cdots q_s$ for some primes p_i and q_j. Therefore

$$k+1 = a \cdot b = p_1 p_2 \cdots p_r \, q_1 q_2 \cdots q_s$$

is a product of primes, so $P(k+1)$ holds.
In both cases $P(k+1)$ holds.

Conclusion. By the principle of strong induction, every integer $n \ge 2$ is a product of primes. ■

Cases are natural here. In Case 2 you must check that a and b are inside the range *of the hypothesis ($2 \le a, b \le k$) before applying it; that is what the short inequality does.*

3.3 The actual explanation: the principle and how to write it

Principle of Strong Induction

Let $P(n)$ be a statement about the integer n, and let $n_0 \le n_1$ be integers. Suppose that
(i) **(base cases)** $P(n_0), P(n_0+1), \ldots, P(n_1)$ are all true, and
(ii) **(inductive step)** for every integer $k \ge n_1$: if $P(n_0), P(n_0+1), \ldots, P(k)$ are all true, then $P(k+1)$ is true.
Then $P(n)$ is true for every integer $n \ge n_0$.

Two things changed compared with ordinary induction. The **inductive hypothesis** is now "$P(j)$ holds for all j with $n_0 \le j \le k$" (all previous cases), and there may be **several base cases** $n_0, \ldots, n_1$. The arbitrary k in the step starts at the *last* base case n_1 (so that the first use of the step, $k = n_1$, proves $P(n_1+1)$). When one base case suffices, $n_1 = n_0$.

Remark: strong induction is not "stronger" (optional)

Strong induction proves nothing that ordinary induction cannot; it is the same principle in a more convenient form. Indeed, let $Q(k)$ be the statement "$P(n_0), \ldots, P(k)$ all hold". The base cases give $Q(n_1)$, and the strong inductive step says exactly that $Q(k)$ implies $Q(k+1)$ for $k \ge n_1$. Ordinary induction on Q then gives $Q(n)$ for all $n \ge n_1$, hence $P(n)$ for all $n \ge n_0$. So "strong" refers to the *hypothesis you are allowed to assume*, not to the power of the method. Use whichever form makes the step easy to write.

Writing template for a proof by strong induction

Statement. For every integer $n \ge n_0$, *[the claim about n].*
Solution. Let $P(n)$ be the statement "*[the claim, written with n]*". We prove that $P(n)$ holds for every integer $n \ge n_0$ by strong induction on n.
Base case(s). We verify $P(n_0)$, $P(n_0+1)$, $\ldots$, $P(n_1)$ directly. *[One line each.]*

Inductive hypothesis. Let $k \geq n_1$ be an arbitrary integer and assume that $P(j)$ holds for every integer j with $n_0 \leq j \leq k$.
Inductive step. We must show that $P(k+1)$ holds, that is, *[write out $P(k+1)$]. [Reduce $k+1$ to one or more earlier values j. Check that each such j satisfies $n_0 \leq j \leq k$, then apply the hypothesis to it, saying "by the inductive hypothesis".]* Hence $P(k+1)$ holds.
Conclusion. By the principle of strong induction, $P(n)$ holds for every integer $n \geq n_0$. ■

Phrasebook additions for strong induction

Where	What to write
Setting up	"We proceed by strong induction on n."
Base cases	"Base cases ($n = 12, 13, 14, 15$). We check each directly: ..."
Hypothesis	"Let $k \geq n_1$ be an arbitrary integer and assume that $P(j)$ holds for all integers j with $n_0 \leq j \leq k$." "Assume that the statement is true for all integers from n_0 up to k."
Using it	"Since $n_0 \leq k-3 \leq k$, the inductive hypothesis applies to $k-3$, so ..." "Because $2 \leq a \leq k$, by the inductive hypothesis a is ..." "Applying the inductive hypothesis to k and to $k-1$, ..."
Concluding	"By the principle of strong induction, $P(n)$ holds for all integers $n \geq n_0$."

How many base cases do you need? This is the question students most often get wrong, so here is the rule.

Rule for base cases

Look at the inductive step and ask: *which earlier cases does it use to get $P(k+1)$?*

- If it always reaches back a **fixed number** of steps, say to $P(k+1-s)$, you need s **consecutive** base cases $P(n_0), \ldots, P(n_0+s-1)$, and the hypothesis must start at $k \geq n_0 + s - 1$. (Otherwise, for the first few values, the step would reach below n_0, where nothing has been proved.)
- If it reaches back a **varying** amount (a divisor, roughly half, "some smaller j"), make sure the value you reach is always $\geq n_0$, and check the first few values above the base case by hand.
- Recursively defined sequences: at least one base case for every initial value in the definition.

After writing the proof, test the step with the *smallest* allowed k. If it tries to use a case you have not proved, add base cases.

3.4 Comprehensive examples

Example 6 (comprehensive): postage with 4-cent and 5-cent stamps

Statement. For every integer $n \geq 12$, it is possible to make exactly n cents of postage using only 4-cent and 5-cent stamps.

Scratch work. "n cents can be made" means: there are non-negative integers a and b (numbers of 4-cent and 5-cent stamps) with $n = 4a + 5b$. Try small values: $12 = 4+4+4$, $13 = 4+4+5$, $14 = 4+5+5$, $15 = 5+5+5$, $16 = 4+4+4+4$, $17 = 4+4+4+5$, ...

How would we make $k+1$ cents if we can already make smaller amounts? A natural idea: make

$k+1-4 = k-3$ cents first, then add one 4-cent stamp. This uses the case $k-3$, three steps below k, so ordinary induction (which only offers $P(k)$) is not enough; strong induction is the right tool.

How many base cases? The step reaches back 4 positions (from $k+1$ to $k-3$), so we need 4 consecutive base cases: $12, 13, 14, 15$. The hypothesis then starts at $k \geq 15$, so that $k-3 \geq 12$ is always inside the proved range.

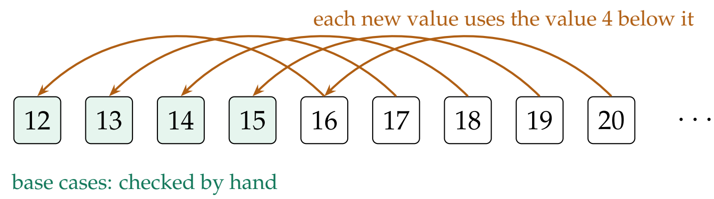

Figure 6. Why four base cases: 16 *needs* 12*,* 17 *needs* 13*,* 18 *needs* 14*,* 19 *needs* 15*. With only* 12 *as a base case, proving* 13 *would require* 9*, which is outside the statement.*

Model proof of Example 6

Proof (what you write)	**Commentary (what it does)**
Statement. For every integer $n \geq 12$, n cents of postage can be made using only 4-cent and 5-cent stamps.	
Solution. Let $P(n)$ be the statement "there exist non-negative integers a, b such that $n = 4a + 5b$". We prove that $P(n)$ holds for every integer $n \geq 12$ by strong induction on n.	*Translate the word problem into a precise statement about integers.*
Base cases ($n = 12, 13, 14, 15$). $12 = 4 \cdot 3 + 5 \cdot 0$, $13 = 4 \cdot 2 + 5 \cdot 1$, $14 = 4 \cdot 1 + 5 \cdot 2$, $15 = 4 \cdot 0 + 5 \cdot 3$. Hence $P(12), P(13), P(14), P(15)$ all hold.	*Four consecutive base cases, because the step reaches back four positions.*
Inductive hypothesis. Let $k \geq 15$ be an arbitrary integer and assume that $P(j)$ holds for every integer j with $12 \leq j \leq k$.	*k starts at the* last *base case, 15.*
Inductive step. We must show that $P(k+1)$ holds. Since $k \geq 15$, we have $12 \leq k-3 \leq k$, so the inductive hypothesis applies to $k-3$: there are non-negative integers a, b with $k-3 = 4a+5b$. Then $$k+1 = (k-3)+4 = 4a+5b+4 = 4(a+1)+5b,$$ and $a+1$ and b are non-negative integers. Hence $P(k+1)$ holds.	*The sentence "$12 \leq k-3 \leq k$" is not decoration: it certifies that $k-3$ is inside the range of the hypothesis.*
Conclusion. By the principle of strong induction, $P(n)$ holds for every integer $n \geq 12$. ■	

Spot the error: the same problem "done wrong"

"Proof." Base case: $12 = 4+4+4$. Inductive step: assume k cents can be made for some $k \geq 12$. To make $k+1$ cents, take the stamps for k cents and replace one 4-cent stamp by a 5-cent stamp. ■

Where is the mistake? (Answer in Appendix A.) Hint: what if the collection for k cents contains no 4-cent stamp, as in $15 = 5+5+5$?

Example 7 (comprehensive): a closed formula for a recursively defined sequence

Define a sequence by $a_1 = 1$, $a_2 = 5$, and $a_n = a_{n-1} + 2a_{n-2}$ for every integer $n \geq 3$.
Statement. For every integer $n \geq 1$, $\quad a_n = 2^n + (-1)^n$.

Scratch work. Compute a few terms from the definition and compare with the formula:

n	1	2	3	4	5
a_n from the definition	1	5	$5 + 2 \cdot 1 = 7$	$7 + 2 \cdot 5 = 17$	$17 + 2 \cdot 7 = 31$
$2^n + (-1)^n$	$2 - 1 = 1$	$4 + 1 = 5$	$8 - 1 = 7$	$16 + 1 = 17$	$32 - 1 = 31$

The recurrence expresses a_{k+1} through the *two* previous terms a_k and a_{k-1}. So the inductive step will use $P(k)$ and $P(k-1)$: we need the strong form of the hypothesis, and *two* base cases ($n = 1$ and $n = 2$, the two initial values). The hypothesis starts at $k \geq 2$, so that $k - 1 \geq 1$.

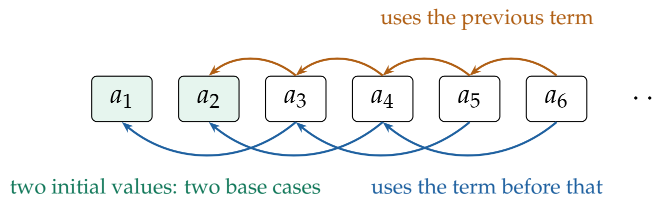

Figure 7. Each term depends on the two before it, so the step needs $P(k)$ and $P(k-1)$, and the first step ($k = 2$) needs both base cases $P(1)$ and $P(2)$.

Model proof of Example 7

Proof (what you write)	**Commentary (what it does)**
Statement. For every integer $n \geq 1$, $a_n = 2^n + (-1)^n$, where $a_1 = 1$, $a_2 = 5$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$.	
Solution. Let $P(n)$ be the statement "$a_n = 2^n + (-1)^n$". We prove that $P(n)$ holds for every integer $n \geq 1$ by strong induction on n.	
Base cases ($n = 1$ and $n = 2$). By definition $a_1 = 1$, and $2^1 + (-1)^1 = 2 - 1 = 1$, so $P(1)$ holds. By definition $a_2 = 5$, and $2^2 + (-1)^2 = 4 + 1 = 5$, so $P(2)$ holds.	*One base case per initial value of the sequence.*
Inductive hypothesis. Let $k \geq 2$ be an arbitrary integer and assume that $P(j)$ holds for every integer j with $1 \leq j \leq k$, that is, $a_j = 2^j + (-1)^j$ for all such j.	*$k \geq 2$ guarantees that $k - 1 \geq 1$ is covered.*

Inductive step. We must show that $P(k+1)$ holds, that is, $a_{k+1} = 2^{k+1} + (-1)^{k+1}$. Since $k+1 \geq 3$, the recurrence applies: $a_{k+1} = a_k + 2a_{k-1}$. Because $1 \leq k-1 \leq k$ and $1 \leq k \leq k$, the inductive hypothesis applies to both k and $k-1$, giving

$$
\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \\
&= \left(2^k + (-1)^k\right) + 2\left(2^{k-1} + (-1)^{k-1}\right) && \text{[by the inductive hypothesis]} \\
&= 2^k + 2^k \; + \; (-1)^k + 2(-1)^{k-1} \\
&= 2^{k+1} \; + \; (-1)^{k-1}\left((-1) + 2\right) && [\text{since } (-1)^k = (-1)^{k-1} \cdot (-1)] \\
&= 2^{k+1} + (-1)^{k-1} \\
&= 2^{k+1} + (-1)^{k+1}, && [\text{since } (-1)^{k+1} = (-1)^{k-1} \cdot (-1)^2]
\end{aligned}
$$

which is precisely $P(k+1)$.

Conclusion. By the principle of strong induction, $a_n = 2^n + (-1)^n$ for every integer $n \geq 1$. ■

First justify that the recurrence may be used ($k+1 \geq 3$), then that both earlier indices are in range. The algebra is routine; write it in small steps so each equality is checkable.

3.5 Common mistakes specific to strong induction

Mistakes that cost marks

1. **Too few base cases.** If the step reaches back s positions, you need s consecutive base cases (Example 6 needs 12 to 15; Example 7 needs $n = 1$ and $n = 2$).
2. **Hypothesis starting too early.** With base cases $n_0, \ldots, n_1$, the arbitrary k must satisfy $k \geq n_1$, not $k \geq n_0$; otherwise the first step reaches below n_0.
3. **Applying the hypothesis out of range.** Before using $P(j)$, check and *say* that $n_0 \leq j \leq k$ ("since $k \geq 15$, we have $12 \leq k-3 \leq k$").
4. **Using the recurrence where it is not defined.** A rule such as $a_n = a_{n-1} + 2a_{n-2}$ "for $n \geq 3$" may only be applied to indices ≥ 3; say why the index you use qualifies.
5. **Vague hypothesis.** "Assume it is true for smaller values" is not a hypothesis. Write the range: "for every integer j with $n_0 \leq j \leq k$".

3.6 Quick recap: strong induction

Strong induction in one breath

- Same goal as ordinary induction: "for every integer $n \geq n_0$, $P(n)$".
- Use it when $P(k+1)$ depends on earlier cases other than (or in addition to) $P(k)$: a divisor, $k-1$ and k, $k-3$, roughly $k/2$, and so on.
- **Base cases:** as many consecutive ones as the step reaches back; test the smallest k.
- **Inductive hypothesis:** let $k \geq n_1$ (last base case) be arbitrary; assume $P(j)$ for all $n_0 \leq j \leq k$.
- **Inductive step:** reduce $k+1$ to earlier values, check they lie in $[n_0, k]$, apply the hypothesis.
- **Conclusion:** "by the principle of strong induction, $P(n)$ holds for all $n \geq n_0$."
- It is never wrong to assume the strong hypothesis, even if you end up using only $P(k)$.

4 Putting it together

4.1 Ordinary versus strong induction, side by side

	Ordinary induction	**Strong induction**
Goal	$P(n)$ for all integers $n \geq n_0$	$P(n)$ for all integers $n \geq n_0$
Base case(s)	$P(n_0)$	$P(n_0), \ldots, P(n_1)$: as many consecutive ones as the step needs
Inductive hypothesis	Let $k \geq n_0$ be arbitrary; assume $P(k)$	Let $k \geq n_1$ be arbitrary; assume $P(j)$ for every j with $n_0 \leq j \leq k$
Inductive step shows	$P(k+1)$	$P(k+1)$
Typical use	sums, products, inequalities, divisibility, "adding one more object"	factorisation, recursively defined sequences, algorithms that recurse on smaller inputs, coin/stamp problems
Concluding sentence	"by the principle of mathematical induction"	"by the principle of strong induction"
Relation	Equivalent in power. Strong induction simply lets you assume more in the step. If in doubt, assume the strong hypothesis.	

4.2 Checklist before you hand in an induction proof

Checklist

- ☐ The **statement** is written precisely, with the range of n ("for every integer $n \geq n_0$").
- ☐ $P(n)$ is **defined** in one sentence, and the method is announced: "by (strong) induction on n".
- ☐ The **base case(s)** are the first value(s) in the statement and are verified by computing, not by restating the claim.
- ☐ The **hypothesis** fixes one arbitrary k in the correct range and writes out what is assumed.
- ☐ The **step** states the target $P(k+1)$ explicitly, uses the hypothesis (labelled "by the inductive hypothesis"), and checks that every earlier case used is inside the proved range.
- ☐ The step ends with "which is precisely $P(k+1)$" and the proof ends with the **concluding sentence** and ■.
- ☐ The step works for *every* allowed k, including the smallest one (test it!).

4.3 Exercises

Write each proof in full, using the template. Hints and solutions are in Appendix B.

Ordinary induction.

E1. For every integer $n \geq 1$, $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

E2. For every integer $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

E3. For every integer $n \geq 1$, $n^3 - n$ is divisible by 3.

E4. For every integer $n \geq 4$, $n! > 2^n$.

E5. For every integer $n \geq 1$, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Strong induction.

E6. The Fibonacci numbers are $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that $F_n < 2^n$ for every integer $n \geq 0$.

E7. Prove that every amount of postage of 8 cents or more can be made with 3-cent and 5-cent stamps.

E8. Let $c_0 = 2$, $c_1 = 5$ and $c_n = 5c_{n-1} - 6c_{n-2}$ for $n \geq 2$. Prove that $c_n = 2^n + 3^n$ for every $n \geq 0$.

E9. Prove that every integer $n \geq 1$ can be written as a sum of *distinct* powers of 2 (for example $13 = 8 + 4 + 1$). This says that binary representations exist.

E10. (Ordinary induction can do it too.) Prove the statement of Example 6 by *ordinary* induction. Hint: in the step, split into two cases according to whether the collection for k cents contains a 4-cent stamp.

A Answers to the "spot the error" puzzles

Horses (Section 2.5). The inductive step is wrong for $k = 1$. With $k + 1 = 2$ horses, the two groups are $\{h_1\}$ and $\{h_2\}$, which have *no horse in common*, so nothing forces h_1 and h_2 to share a colour. The step "$P(1)$ implies $P(2)$" is therefore not proved (indeed $P(2)$ is false), and the whole chain breaks at its first link. The argument does work for $k \geq 2$, which is exactly why it looks convincing. Lesson: the step must be valid for *every* $k \geq n_0$, including the smallest one.

Stamps done wrong (Section 3.4). The step assumes that the collection for k cents contains a 4-cent stamp. For $k = 15$ (and 20, 25, . . .) it may not: $15 = 5 + 5 + 5$. So the step fails at $k = 15$ and $P(16)$ is never reached by this argument. Two correct repairs: use strong induction with base cases 12 to 15 (Example 6), or keep ordinary induction and add the missing case (Exercise E10).

B Hints and solutions to the exercises

Only the mathematical core of each proof is given; you should wrap it in the full template.

E1. $P(n)$: $1 + 3 + \cdots + (2n-1) = n^2$. Base ($n = 1$): $1 = 1^2$. Step: assume $P(k)$ for some $k \geq 1$; then $1 + 3 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1) = (k+1)^2$, which is $P(k+1)$.

E2. Base ($n = 0$): $\sum_{i=0}^{0} 2^i = 1 = 2^1 - 1$. Step: assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ for some $k \geq 0$; then $\sum_{i=0}^{k+1} 2^i = (2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$.

E3. Base ($n = 1$): $1^3 - 1 = 0 = 3 \cdot 0$. Step: assume $k^3 - k = 3q$ for some integer q, where $k \geq 1$. Then $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + 3(k^2 + k) = 3q + 3(k^2 + k) = 3\,(q + k^2 + k)$, and $q + k^2 + k$ is an integer.

E4. Base ($n = 4$): $4! = 24 > 16 = 2^4$. Step: assume $k! > 2^k$ for some $k \geq 4$. Then $(k+1)! = (k+1) \cdot k! > (k+1) \cdot 2^k \geq 2 \cdot 2^k = 2^{k+1}$, because $k + 1 \geq 5 \geq 2$.

E5. Base ($n = 1$): $1^2 = 1$ and $\frac{1 \cdot 2 \cdot 3}{6} = 1$. Step: assume the formula for some $k \geq 1$. Then

$$
\begin{aligned}
1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)\big[k(2k+1) + 6(k+1)\big]}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6},
\end{aligned}
$$

which is the formula with $n = k + 1$ (note $(k+2)(2k+3) = 2k^2 + 7k + 6$).

E6. Two base cases: $F_0 = 0 < 1 = 2^0$ and $F_1 = 1 < 2 = 2^1$. Hypothesis: let $k \geq 1$ and assume $F_j < 2^j$ for all $0 \le j \le k$. Step: since $k+1 \ge 2$ the recurrence applies, and $k-1 \ge 0$, so $F_{k+1} = F_k + F_{k-1} < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$.

E7. The step will add a 3-cent stamp to the collection for $k-2$, reaching back 3 positions, so three base cases: $8 = 3+5$, $9 = 3+3+3$, $10 = 5+5$. Hypothesis: let $k \ge 10$ and assume $P(j)$ for all $8 \le j \le k$. Step: since $k \ge 10$, $8 \le k-2 \le k$, so $k-2 = 3a+5b$ for some non-negative integers a, b, and then $k+1 = 3(a+1)+5b$.

E8. Base cases: $c_0 = 2 = 2^0 + 3^0$ and $c_1 = 5 = 2^1 + 3^1$. Hypothesis: let $k \ge 1$ and assume $c_j = 2^j + 3^j$ for all $0 \le j \le k$. Step: $k+1 \ge 2$, so $c_{k+1} = 5c_k - 6c_{k-1} = 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1}) = 5 \cdot 2^k - 3 \cdot 2^k + 5 \cdot 3^k - 2 \cdot 3^k = 2 \cdot 2^k + 3 \cdot 3^k = 2^{k+1} + 3^{k+1}$ (using $6 \cdot 2^{k-1} = 3 \cdot 2^k$ and $6 \cdot 3^{k-1} = 2 \cdot 3^k$).

E9. Base ($n = 1$): $1 = 2^0$. Hypothesis: let $k \ge 1$ and assume every integer j with $1 \le j \le k$ is a sum of distinct powers of 2. Step: let 2^m be the largest power of 2 with $2^m \le k+1$, and let $r = (k+1) - 2^m$. Then $0 \le r \le k$, and $r < 2^m$ (otherwise $2^{m+1} \le k+1$, contradicting the choice of m). If $r = 0$, then $k+1 = 2^m$ and we are done. Otherwise $1 \le r \le k$, so by the hypothesis r is a sum of distinct powers of 2; each of them is at most $r < 2^m$, so adding 2^m keeps the powers distinct, and $k+1 = 2^m + r$ is the required sum.

E10. $P(n)$: $n = 4a + 5b$ for some non-negative integers a, b. Base ($n = 12$): $12 = 4 \cdot 3$. Step: assume $k = 4a + 5b$ for some $k \ge 12$. *Case $a \ge 1$:* $k+1 = 4(a-1) + 5(b+1)$. *Case $a = 0$:* then $k = 5b \ge 12$ forces $b \ge 3$, and $k+1 = 5b+1 = 5(b-3) + 16 = 4 \cdot 4 + 5(b-3)$ with $b - 3 \ge 0$. In both cases $P(k+1)$ holds.

Sources consulted in preparing these notes: MIT 6.1200J lecture notes on induction and strong induction (Abel, Chapman, Demaine, 2024); J. Erickson, Algorithms*, Appendix I "Proof by Induction" (U. Illinois); CSE 311 (U. Washington) lecture on strong induction; CSI2101 (U. Ottawa, L. Moura) induction slides; COMP 250 (McGill) induction slides; CSC110/111 course notes, "Proof by Induction" (U. Toronto); MATH 2420 lecture notes on induction (GeorgiaTech); RGASC handout on mathematical and strong induction (U. Toronto).*